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Interim Report

MATHEMATICAL BACKGROUND

by

H. L. Saxton

Submitted to

Naval Ship Systems Command

Department of the Navy

Washington, D. C. 20360

Attn: Code 00V1B

Expanded and Resubmitted on

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TRACOR

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CHAPTER I

MATHEMATICAL BACKGROUND

USE OF THE DECIBEL SCALE

In this report gains and losses will be expressed in decibels, in conformance with common practice in the field of communications. The "bel" is named after Alexander Graham Bell, who invented the telephone. It is defined as the logarithm to base 10 of an energy ratio. Since the bel is too large a unit for greatest convenience, the use of the decibel is accepted even though values may get to be over 100 decibels. The definition, then, is

$$\text{decibels (db)} = 10 \log \frac{E_2}{E_1} \qquad (1.1)$$

where frequently E_2 is output energy and E_1 is input energy.

Considerable liberty has been taken with the use of the decibel scale without much regard for the strict definition. In this report power and intensity will be used more than energy, and there will be no hesitancy to express a power ratio or an intensity ratio in decibels by taking the logarithm and multiplying by 10. Furthermore, an absolute power or intensity level will be expressed in decibels with respect to an accepted reference level. The reference level for power is accepted as one watt. That of intensity is controversial. Since standardization is pending at this time, a reference level for intensity acceptable to most of our readers will be chosen, viz, the power associated with a pressure level of one

(level of one) microbar when the pressure is operating into a purely resistive load.

The practice of expressing voltage or current gains or losses in decibels has sometimes been done incorrectly. We shall consider voltage gain in an amplifier. Suppose the input voltage is e_i across a resistance R_i and the output voltage is e_o across a resistance R_o. Then power input is e_i^2/R_i and power output is e_o^2/R_o so that gain in db is correctly expressed as $10 \log \frac{P_o}{P_i} = 10 \log \frac{e_o^2 R_i}{e_i^2 R_o}$. Now, only if $R_i = R_o$ can the R's be cancelled, and only for this case the expression becomes $20 \log \frac{e_o}{e_i}$. On the other hand, in this example it is true that the general correct expression may be written $20 \log \frac{e_o}{e_i} + 10 \log \frac{R_i}{R_o}$. While neither term by itself conforms in general to the definition of decibels, the two together do conform, and no harm is done by referring to each term as "decibels" as long as both terms are present. However, we must be sure that both are present unless the second one is 0.

An example in which considerable liberty is taken without in any way invalidating the numerical results is in some terms of the echo ranging equation. We may write spherical divergence loss as $20 \log \frac{R_o}{R_i}$ in which R_o and R_i are output and input ranges from a point source. This expression is equivalent to 10 log ratio of powers at the two ranges only if divergence loss is total loss. Therefore, due consideration must be given to other losses to assure a numerically correct result. This is regularly done by introducing the remaining loss terms, also in "decibels."

There is one common situation in which the values of the input and output resistances of an amplifier may be ignored in computing gain in db. This is when our interest is in gain in signal-to-noise ratio. Here the ratio of signal-to-noise powers at the input is equal to the ratio of

(the ratio of) the squares of the signal voltage and the noise voltage because both appear across the same input resistance. Likewise, at the output of the network, resistance is not a factor in the ratio of signal to noise. If the ratio of signal-to-noise voltages at the input were γ_i and at the output were γ_o, then gain is correctly expressed as 20 log $\frac{\gamma_o}{\gamma_i}$. Let it be consciously recognized that this gain is in signal-to-noise ratio.

OPTIMUM VALUES

In underwater sound theory it is frequently desirable to optimize some function by the choice of the value of a controllable variable. This variable is often frequency. Every term in the function may be expressible in terms of frequency. Then again, it may be desirable to hold some of the terms constant regardless of what frequency is chosen. An example of this is found in determining an echo excess where there is a dependence on directivity index. The particular condition of interest may involve a directivity index that is a function of frequency by using a transducer of fixed size, or the condition may involve a directivity index that is kept constant by doubling the transducer dimensions whenever the frequency is halved. In either case, the term in question may be considered as a constant or as a function of the variable as desired in order to obtain results under the desired restrictions.

The mathematical procedure, then, is to differentiate the function with respect to the variable in order to obtain the slope and to equate this slope to 0, which is characteristic of an optimum point. By this procedure, for example, we may solve for the optimum frequency at any range. In general, there is a best frequency for a given acoustic path and a given range.

(a given range.)

STATISTICS

General

Whenever the processes with which we are dealing fluctuate, thereby introducing an element of uncertainty, the most rigorous mathematical treatment is a statistical treatment in which we deal with probabilities of different results.

Functions with Discrete Values

A probability of the occurrence of an event is the ratio of number of occurrences to number of independent trials when the number of trials is very large. It is thus the average number of successes per trial when the number of trials is very large. Suppose an event has a probability $p = .4$ of occurrence. Out of ten trials we would expect, on the average, four occurrences. By "on the average" is meant that after ten trials we might take another set of ten, and another, and another, and we might average the results. We might get, for example, on successive trials, three, four, two, five, four, respectively. On the first set of ten we would judge the probability to be 3/10 or .3. On five sets we would judge the probability to be $(3 + 4 + 2 + 5 + 4)/50 = .36$ so that we would be getting closer to the true value of .4. We might have chanced to hit the probability .4 on the nose on the first trial, but we wouldn't be sure that we had hit it. Our confidence in the result should increase with more trials.

A fundamental law of probability relates to the joint probability of two or more independent events. This joint probability is the product of the individual probabilities. This may be expressed as $P = p_1 p_2 p_3 \dots p_n$. Let us illustrate

(Let us illustrate) this by the case of two independent events. For simplicity, we shall consider the probability of two heads in two tosses of an unweighted coin. The following sequences are all of the possibilities, and each is equally likely: H--H, H--T, T--H, T--T. Since these cases are equally likely, each must have a probability of 1/4. Thus H--H has a probability of 1/4, which is $p_1 p_2 = 1/2 \times 1/2$ and thus illustrates the general rule. We must keep in mind that when we deal with probabilities, we are dealing with average results. We do not say that on just four trials each of the permutations will occur just once. What we do say is that in many sets of four trials, this will be the average result.

The last example yields more information than we sought. It yields the following:

$$P\text{ (two heads)} = .25$$

$$P\text{ (one head, one tail)} = .50$$

$$P\text{ (two tails)} = .25$$

We may ask why one head and one tail is twice as likely as two heads. It is simply that there is only one way of tossing two heads, but there are two ways of tossing one head and one tail, either H on the first toss and T on the second toss of T on the first toss and H on the second. A useful relationship is evidenced here. If we expand $(p + q)^2$, we obtain $p^2 + 2pq + q^2$. Now we let $\rho = .5$ and $q = .5$ so that the three terms in the expansion become .25, .5 and .25. If p and q are respectively the probability of a head and a tail, then each term in the expansion is identifiable with the probability of one of the three combinations.

Now we take a more complicated application of the binomial expansion.

(the binomial expansion.) Suppose a weather prediction has the probability 3/4 of being correct. What is the probability of all combinations of success and failure on three independent predictions? We expand

$$(p + q)^3 = p^3 + 3p^2q + 3pq^2 + q^3$$

Putting $p = 3/4$ and $q = 1/4$, we obtain 27/64, 27/64, 9/64, 1/64, and thus

$$P \text{ (3 successes)} = 27/64$$

$$P \text{ (2 successes only)} = 27/64$$

$$P \text{ (1 success only)} = 9/64$$

$$P \text{ (no successes)} = 1/64$$

We may verify this result by the case method. We may set down sixty-four cases of three predictions in a row. On 3/4 of these the first prediction will be successful and should be so tabulated in the first column, and the remaining 1/4 are unsuccessful and should be so tabulated. Let us say we indicate a success by one and failure by zero. On the second trial 3/4 of the previous successes will be followed by another success, and also 3/4 of the previous failures will be followed by a success. All remaining cases will be failures. Continuing in this way, we may count up the various results and the probabilities just solved for above will be found in agreement.

(found in agreement.)

Bernoulli Distribution

The different coefficients in the binomial expansion represent a number of ways that n things may be taken m at a time. From what we have already observed and/or proved, we may write a formula for m successes out of n trials as follows:

$$P_n(m) = \frac{n!}{m!(n - m)!} p_1^{\,m} q_1^{\,n - m} \qquad (1.2)$$

in which $P_n(m)$ is probability of m successes out of n trials.

p_1 is probability of success on one trial

$q_1 = 1 - p_1$ is probability of a failure on one trial

The values $P_n(m)$ for the various m constitute what is named a "Bernoulli distribution." If we list terms in order starting with m = n and decreasing one at a time, we obtain precisely and in order the terms of $(p_1 + q_1)^n$ expanded.

As an example of the application of the Bernoulli distribution, suppose that we are receiving echoes on a sonar system at a level such that the probability of an echo exceeding a given threshold and producing a mark on each ping is 1/3. Let us find the probability of at least two marks out of five successive pings. Here we have $n = 5$, $p_1 = 1/3$, $q_1 = 2/3$. The simplest way to get this is first to find the probability of just zero or one mark. For these cases we have m = 0 and 1 so that

$$P_5\ (0) = \frac{5!}{0!\ 5!} (1/3)^0\ (2/3)^5 = .132$$

$$P_5\ (1) = \frac{5!}{1!\ 4!} (1/3)\ (2/3)^4 = \underline{.329}$$

$$P_5\ (0) + P_5\ (1) = \underline{.461}$$

(1 so that)

$$P_5\ (m < 2) = .461.$$

Now the probability of $P_5\ (m \geqslant 2) = 1 - .461 = .539$.

The preceding example was for a signal present, and it will be noted that $P_5\ (m > 2) > p_1$. It is now assumed that the signal was in combination with noise. Suppose now that we have noise alone with the probability of exceeding threshold at the range of the target of .01, that is, $p_1 = .01$. What is the probability of two out of five marks at this particular range on noise alone? The same steps as before give $p_5\ (m > 2) = .001$. So we see that while the probability of detection was improved by requiring two out of five rather than one out of one, the probability of false alarm was decreased by an order of magnitude. We should certainly expect to gain something by using five times as much time (five pings instead of one), and in this case the gain is apparent.

It is now asserted that when n is large, the binomial distribution is approximately given by

$$P_n(m) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(m - \bar{m})}{2\sigma^2}} \tag{1.3}$$

in which $\sigma^2 = npq$, $\bar{m} = np$, n is number of independent trials, p is probability of success in any single trial, $\bar{m}$ is average number of successes in n trials, and m is any given number of successes for which the probability is desired. σ is just a convenient symbol for the time being.

We now take a simple example. We shall attempt to find the probability

(find the probability) of $m = \overline{m}$ in ten trials when $p = .1$. For this case $\sigma = \sqrt{10\ (.1)\ (.9)} = .95$, $\overline{m} = np = 10\ (.1) = 1$, the exponential term is unity, and we have $\frac{1}{\sigma\sqrt{2\pi}} = .42$. The true value is given by the binomial distribution formula

$$p_{10}\ (1) = .39.$$

There is a deviation of about 8% of .42 from .39. The central part of the curve, or any part, does better when n is larger. For instance, change p in the above example to .5 so that $\overline{m} = 5$ and $\sigma = 1.58$. Then we have $\frac{1}{\sigma\sqrt{2\pi}} = .252$. We compare this with $p_{10}\ (5) = .247$. The deviation in this case is about 2%.

The two cases worked out above are plotted for all integral values of m in Fig 1.1 along with an example in which n is small and the approximation is correspondingly slightly poorer. The approximate form is plotted as discrete values connected by a smooth curve. If in Eq. (1.3) we replace $m - \overline{m}$ by $r - \mu$ and let r vary continuously instead of taking on discrete values, Eq. (1.3) becomes

$$p\ (r) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{(r-\mu)^2}{2\sigma^2}} . \qquad (1.4)$$

This is the Guassian probability density function.

Guassian PDF and Distribution

With a random variable having a continuous amplitude distribution, p (r) dr is defined as the probability that the value of the variable lies between r and r + dr. The p (r) is called a probability density function (PDF). A Guassian distribution may be defined by its PDF,

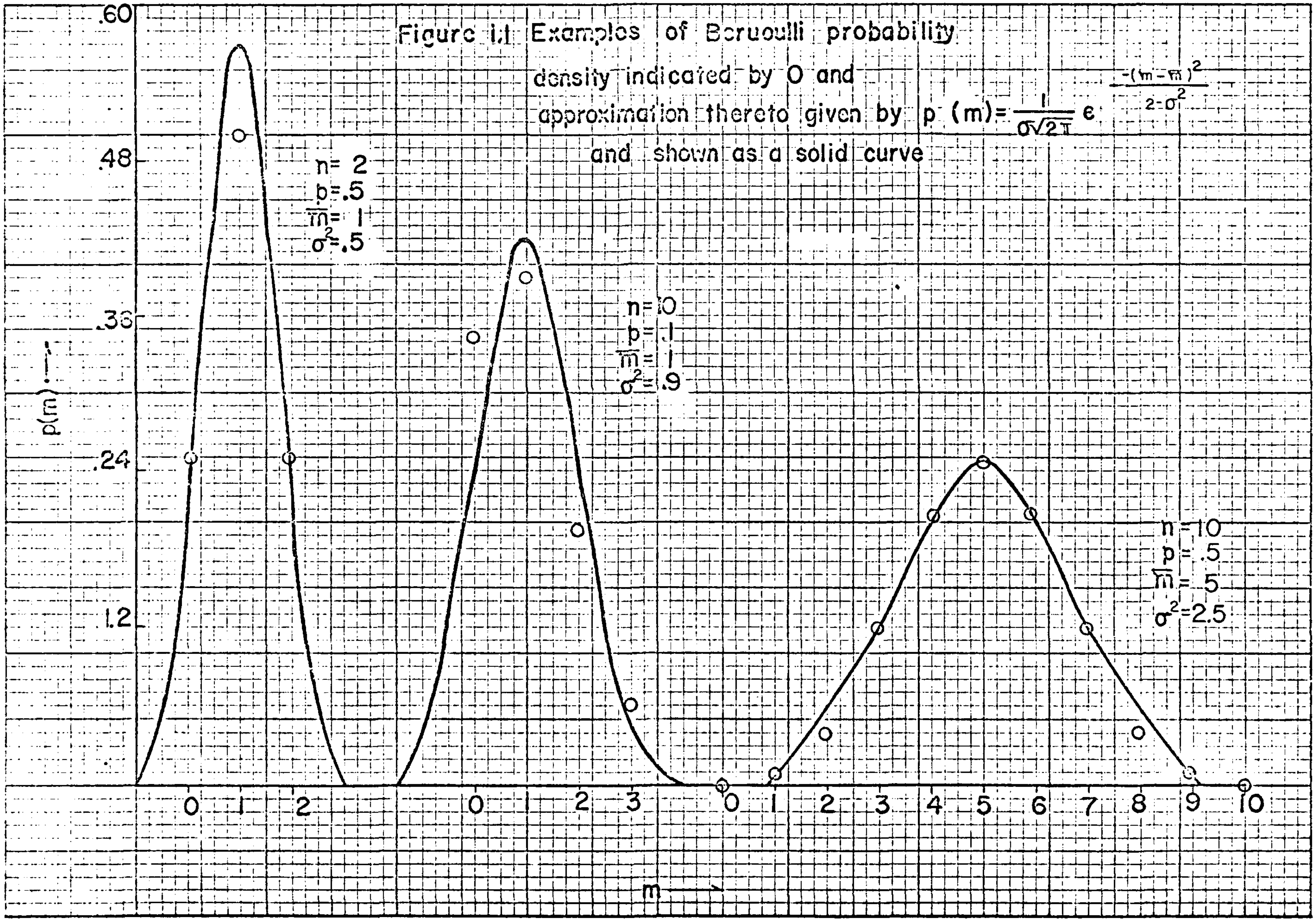


Figure 1.1 Examples of Bernoulli probability density indicated by O and approximation thereto given by $p(m) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(m-\overline{m})^2}{2\sigma^2}}$ and shown as a solid curve

$$p(r) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2 \frac{(r-\mu)^2}{\sigma^2}} \quad -\infty < r < \infty. \tag{1.5}$$

A graph of the PDF with the variable expressed as $r' = \frac{r-\mu}{\sigma}$ is given in Fig. 2.

1,2 —

The nth moment of a variable r with a PDF given by p (r) is defined as

$$M_n = \int_{-\infty}^{\infty} r^n p\,(r)\, dr\,. \tag{1.6}$$

The zeroth moment is unity provided that the coefficient has been correctly specified. The value of unity will now be derived for the PDF of Eq. (1.5) by direct integration:

$$M_0 = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(r-\mu)^2}{2\sigma^2}} dr. \tag{1.7}$$

We let

$$x = \left(\frac{r-\mu}{\sigma}\right)$$

and

$$dr = \sigma dx.$$

Substituting Eq. (1.8) in Eq. (1.7), we obtain

$$M_0 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx. \tag{1.8}$$

The same form of expression

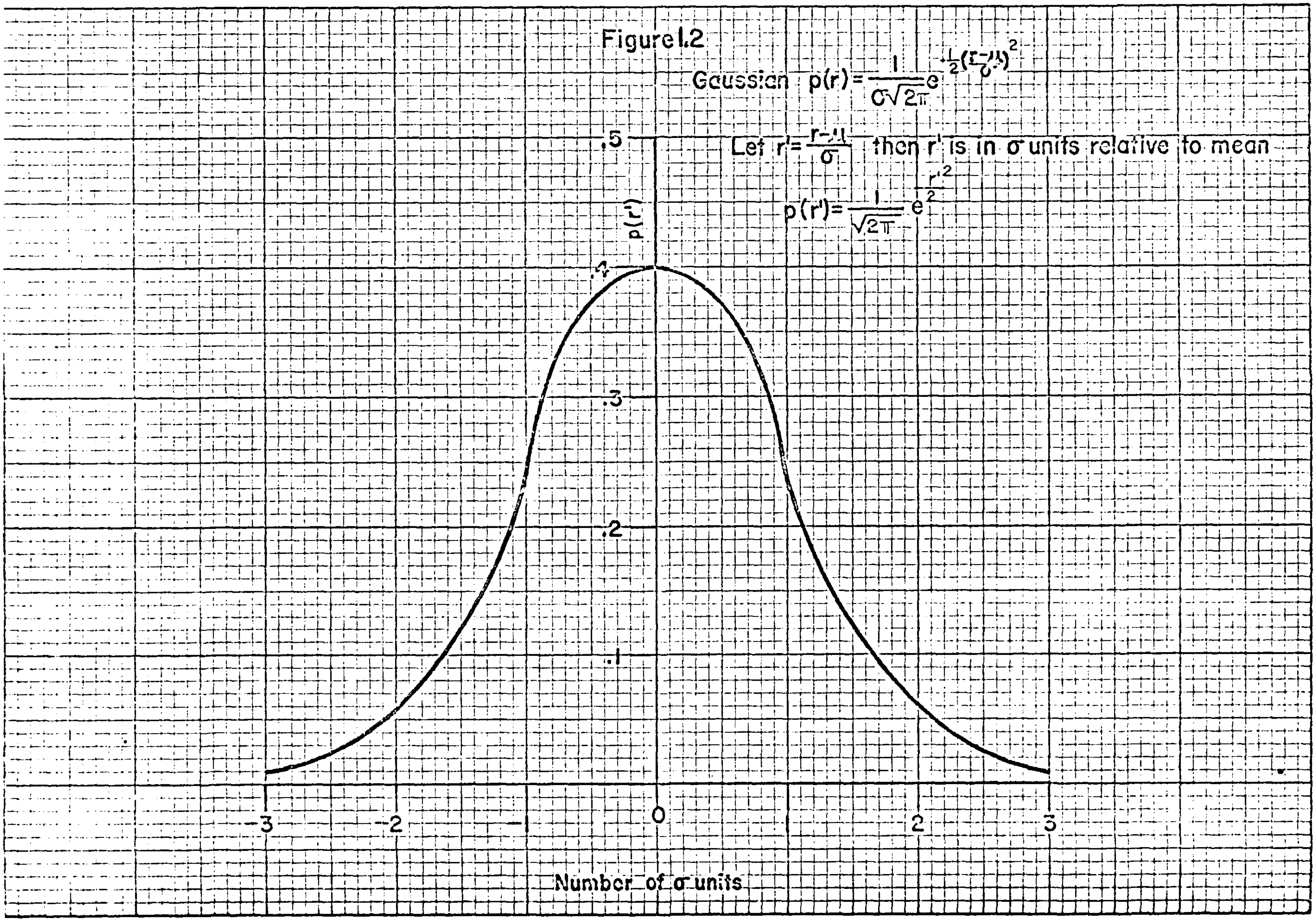


Figure 1.2
Gaussian $p(r)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2}$
Let $r'=\frac{r-\mu}{\sigma}$ then r' is in σ units relative to mean
$p(r')=\frac{1}{\sqrt{2\pi}}e^{-\frac{r'^2}{2}}$
p(r')
.5
.4
.3
.2
.1
-3
-2
0
2
3
Number of σ units

(form of expression) in Eq. (1.8) would hold if the variable were y. The value M_0 may therefore be expressed as

$$M_0 = \sqrt{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{x^2 + y^2}{2}} dx\, dy}. \tag{1.9}$$

In Eq. (1.9) the integral is over all the area in a plane. We may change to polar coordinates for which $r^2 = x^2 + y^2$ and $dx\, dy = r\, d\, r\, d\, \theta$ and then rewrite Eq. (1.9) as

$$M_0 = \sqrt{\int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{\infty} e^{-\frac{r^2}{2}} r\, dr} \quad . \tag{1.10}$$

The integral with respect to θ is 1. The integral with respect to r is $- e^{-\frac{r^2}{2}} \Big|_0^{\infty} = 1$ and therefore $M_0 = 1$. QED.

The first moment is the integral of the variable weighted by the probability density function and is therefore the average value of the function. For the Guassian distribution this is

$$M_1 = \int_{-\infty}^{\infty} \frac{r}{\sigma\sqrt{2\pi}} e^{-1/2 \frac{(r - \mu)^2}{2}} dr. \tag{1.11}$$

Letting $r' = \frac{r - \mu}{\sigma}$, we have $dr = \sigma\, dr'$. Substituting and dividing M_1 into two integrals, we obtain

$$M_1 = \int_{-\infty}^{+\infty} \frac{r'\sigma}{\sqrt{2\pi}} e^{-\frac{r'^2}{2}} dr' + \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{r'^2}{2}} dr. \tag{1.12}$$

The first integral is that of an odd function and therefore integrates to zero. The second integral is the same as M_0, already evaluated as unity. Multiplying by its coefficient μ, we obtain

(μ, we obtain)

$$M_1 = \mu. \tag{1.13}$$

The second moment is most readily computed when $\mu = 0$, and we shall treat this case. The function M_2 is

$$M_2 = \int_{-\infty}^{+\infty} \frac{r^2}{\sigma\sqrt{2\pi}} e^{-\frac{r^2}{2\sigma^2}} dr. \tag{1.14}$$

We let

$$u = \frac{\sigma r}{\sqrt{2\pi}} \qquad dv = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr$$

and then

$$du = \frac{\sigma dr}{\sqrt{2\pi}} \qquad v = - e^{-\frac{r^2}{2\sigma^2}}$$

$$\int udv = uv - \int vdu$$

$$uv \Big|_{-\infty}^{+\infty} = 0$$

$$- \int_{-\infty}^{\infty} vdu = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{r^2}{2\sigma^2}} dr.$$

We now have

$$M_2 = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2\sigma^2}} dr. \tag{1.15}$$

(We now have)

This is simply σ^2 times M_0 given by Eq. (1.7) when $\mu = 0$. Therefore, for $\mu = 0$ we obtain

$$M_2 = \sigma^2 \tag{1.16}$$

The second moment about the mean is called the variance. If M_2 is computed about 0 when the mean is μ, $\sigma^2 = M_2 - \mu^2$. The symbol $\sigma = +\sqrt{\sigma^2}$ is called the standard deviation.

The distribution function of a random variable ϕ (r) is defined as the probability of the variable not exceeding a specified value of r. This is

$$\phi\ (r) = \int_{-\infty}^{r} p\ (r)\ dr \tag{1.17}$$

and for the Guassian distribution

$$\phi\ (r) = \int_{-\infty}^{r} \frac{1}{\sigma\sqrt{2\pi}}\ e^{-1/2 \left(\frac{r-\mu}{\sigma}\right)^2}\ dr \tag{1.18}$$

In tabulating ϕ it is convenient to introduce a new variable

$$r' = \frac{r-\mu}{\sigma}\ . \tag{1.19}$$

In changing the scale by a factor $\frac{1}{\sigma}$, we must multiply the new PDF by σ since $dr = \sigma dr'$, obtaining

$$p\ (r') = \sigma \frac{1}{\sigma\sqrt{2\pi}}\ e^{-1/2\ r'^2} = \frac{1}{\sqrt{2\pi}}\ e^{-1/2\ r'^2} \tag{1.20}$$

(dr = σdr$'$, obtaining)

Eq. (1.20) is the standard form of the PDF and may be interpreted as its value with zero mean and unity variance. Another good interpretation is that r' is number of σ units of deviation from the mean. In transformations of this kind the factor, in this case σ, is always derivable by taking the derivative of the old variable with respect to the new, in this case from Eq. (1.19). Now the distribution function is given by

$$\phi\,(r') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r'} e^{-1/2\, r'^2}\, dr' \qquad (1.21)$$

The functions p (r') and ϕ (r') are evaluated in oft repeated tables (see Reference 1, p. 167). In Fig.1.2 p(r') is plotted, and in Fig.1.3, 1 - ϕ (r') is plotted. The form in Fig.13 is most useful for probability of exceeding a threshold. To get ϕ (r) for any given r, we first use Eq. (1.19) to get r', and then we use Fig.13 or tables. We may note that

$$1 - \phi\,(-r') = \phi\,(r') \qquad (1.22)$$

For fairly large r' the tail of the distribution function is approximated by

$$1 - \phi\,(r') \sim \frac{1}{\sqrt{2\pi}\, r'}\, e^{-1/2\, r'^2} = \frac{p\,(r')}{r'} \qquad (1.23)$$

For example, when r = 4, the approximation gives .000033, and the true value is .000032, according to tables.

Rayleigh Distribution

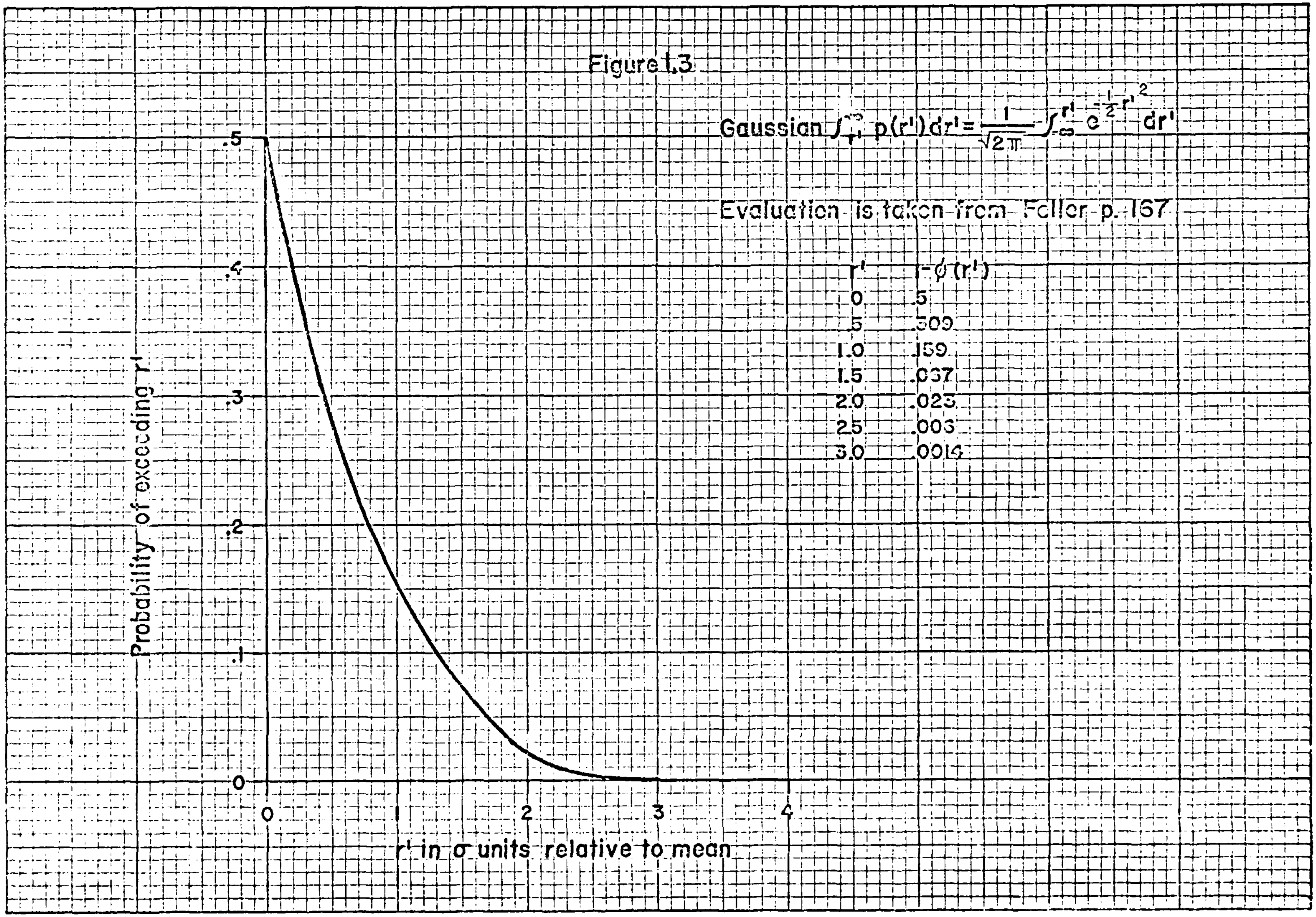
This is the probability associated with a vector in phase space having a certain magnitude when the x and y components are independent and both have Gaussian distributions. It is applicable to an envelope detector. It also turns out to be the kind of distribution we have with a small Bt product (Bt = 1) after

Figure 1.3

Gaussian $\int_{r'}^{\infty} p(r')\,dr' = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r'} e^{-\frac{1}{2}r'^2}\,dr'$

Evaluation is taken from Feller p. 167

r'	$1-\phi(r')$
0	.5
.5	.309
1.0	.159
1.5	.037
2.0	.023
2.5	.003
3.0	.0014



(product (Bt = 1) after)

a linear detector. In contrast to Guassian distribution, the Rayleigh PDF is skewed toward high values and always is positive.

Its derivation is from two Guassian distributions in phase space representing the x and y components of a rotating vector. The density function is the product of their density functions, giving

$$p\ (x,\ y) = \frac{1}{2\pi\sigma^2}\, e^{\frac{-(x^2 + y^2)}{2\sigma^2}} . \tag{1.24}$$

Changing to polar coordinates, we obtain

$$p\ (r,\ \theta) = \frac{r}{2\pi\sigma^2}\, e^{-\frac{r^2}{2\sigma^2}} . \tag{1.25}$$

The r in Eq. (1.25) is the Jacobian,

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r. \tag{1.26}$$

Because of independence from θ, we obtain

$$p\ (\theta) = \frac{1}{2\pi} \tag{1.27}$$

and

$$p\ (r) = \frac{r}{\sigma^2}\, e^{-\frac{r^2}{2\sigma^2}} . \tag{1.28}$$

We replace σ by α since this quantity is no longer standard deviation and write

(deviation and write)

$$p(r) = \frac{r}{\alpha^2} e^{-\frac{r^2}{2\alpha^2}}. \tag{1.29}$$

This integrates readily to give probability of being below r, that is, the "distribution function," as shown below:

$$\phi(r) = \int_0^r p(r)\,dr = -e^{-\frac{r^2}{2\alpha^2}} \Big|_0^r = 1 - e^{-\frac{r^2}{2\alpha^2}}. \tag{1.30}$$

The first moment is

$$M_1 = \mu = \int_0^\infty \frac{r^2}{\alpha^2} e^{-\frac{r^2}{2\alpha^2}}\,dr. \tag{1.31}$$

This integral is that of Eq. (1.14) with α substituted for σ and the new expression multiplied by $\sqrt{2\pi}/2\alpha$, the 2 in denominator taking care of the different limits of integration. Therefore, the result is that of Eq. (1.16) with the substitution of α for σ and the multiplication by $\sqrt{2\pi}/2\alpha$ giving

$$M_1 = \mu = \alpha\sqrt{\frac{\pi}{2}} = 1.26\ \alpha. \tag{1.32}$$

We take the second moment about r = 0

$$M_2 = \int_0^\infty \frac{r^3}{\alpha^2} e^{-\frac{r^2}{2\sigma^2}}\,dr. \tag{1.33}$$

We let

$$u = r^2 \qquad dv = \frac{r}{\alpha^2} e^{-\frac{r}{2\alpha^2}}\,dr$$

(We let)

$$du = 2rdr \qquad v = -e^{-\frac{r^2}{2\alpha^2}}$$

$$-r^2 e^{-\frac{r^2}{2\alpha^2}}\Big|_0^\infty + 2\int_0^\infty re^{-\frac{r^2}{2\alpha^2}}dr = -2\alpha^2 e^{-\frac{r^2}{2\alpha^2}}\Big|_0^\infty = 2\alpha^2$$

$$\sigma^2 = \text{second moment about the mean} = \alpha^2 \quad (2 - \frac{\pi}{2}) = .43\alpha^2 \tag{1.34}$$

$$\sigma = .656\ \alpha. \tag{1.35}$$

Substituting α from (1.35) into (1.32), we obtain

$$\mu = 1.92\ \sigma. \tag{1.36}$$

Having determined μ and σ, we now put p (r)[see Eq. (1.29)] in a form better adapted for plotting. Let

$$r = \sigma r' \tag{1.37}$$

$$J = dr/dr' = \sigma \tag{1.38}$$

$$p\ (r') = \frac{\sigma^2}{\alpha^2} r' e^{-\frac{(\sigma r')^2}{2\alpha^2}} \tag{1.39}$$

Using the relationship between σ and α in Eq. (1.35), we obtain

$$p\ (r') = .43\ r'\ e^{-\ .215\ r'^2} \tag{1.40}$$

This is plotted in Fig. 1.4. In order to use this curve, we determine r' for the given r from Eq. (1.37) and then read p (r') from the curve.

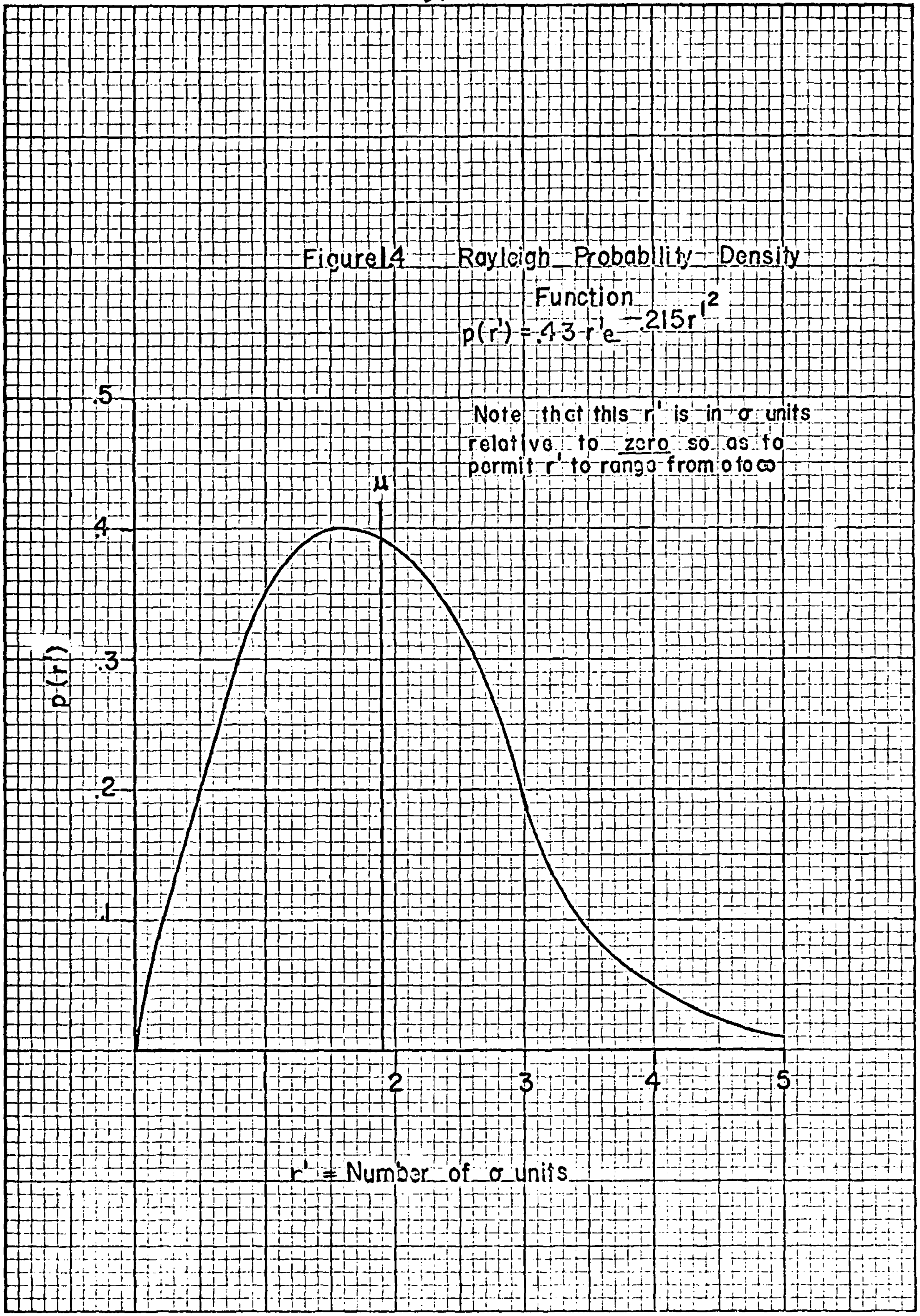


Figure 14 Rayleigh Probability Density Function

$p(r') = .43\,r' e^{-.215 r'^2}$

Note that this r' is in σ units relative to zero so as to permit r' to range from 0 to ∞

(from the curve.)

The function ϕ (r) is modified for plotting only to the extent of using Eq. (1.34) so that Eq. (1.30) becomes

$$1 - \phi (r) = e^{-\frac{.43r^2}{2\sigma^2}} . \qquad (1.41)$$

The graph in Fig.1.5 is of 1 - ϕ (r), probability of exceeding r, for greater convenience than ϕ (r).

Chi Square Distribution

If a number of time series, each having a normal distribution, are squared and then added together, there is obtained a Chi Square PDF. The number of normal functions involved here, n=number of degrees of freedom, affects the particular Chi Square distribution obtained, and we shall therefore use a subscript n to denote the particular distribution involved. When n = 2, the Rayleigh PDF of r is obtained with $p(\chi_2^{\ 2})\, d\chi_2^{\ 2}$ giving $p(r)\, d\chi$ when χ on the right is replaced by r. This has already been covered in some detail. All Chi Square distributions involve a variable squared that is greater than zero. That is, they range from zero to infinity. As n is increased to very large values, e. g. 100, the Chi Square distribution approaches the normal distribution.

The formula for the Chi Square follows:

$$p(\chi_n^{\ 2}) = \left[2^{\ n/2}\, \Gamma\left(\frac{n}{2}\right)\right]^{-1} (\chi^2)^{\left(\frac{n}{2} - 1\right)} e^{-\frac{\chi^2}{2}} . \qquad (1.42)$$

The Γ function of an integer, α, is given by

$$\Gamma(\alpha + 1) = \alpha! . \qquad (1.43)$$

When n is odd, we obtain n/2 as a fraction. In this case, Γ (n/2) is

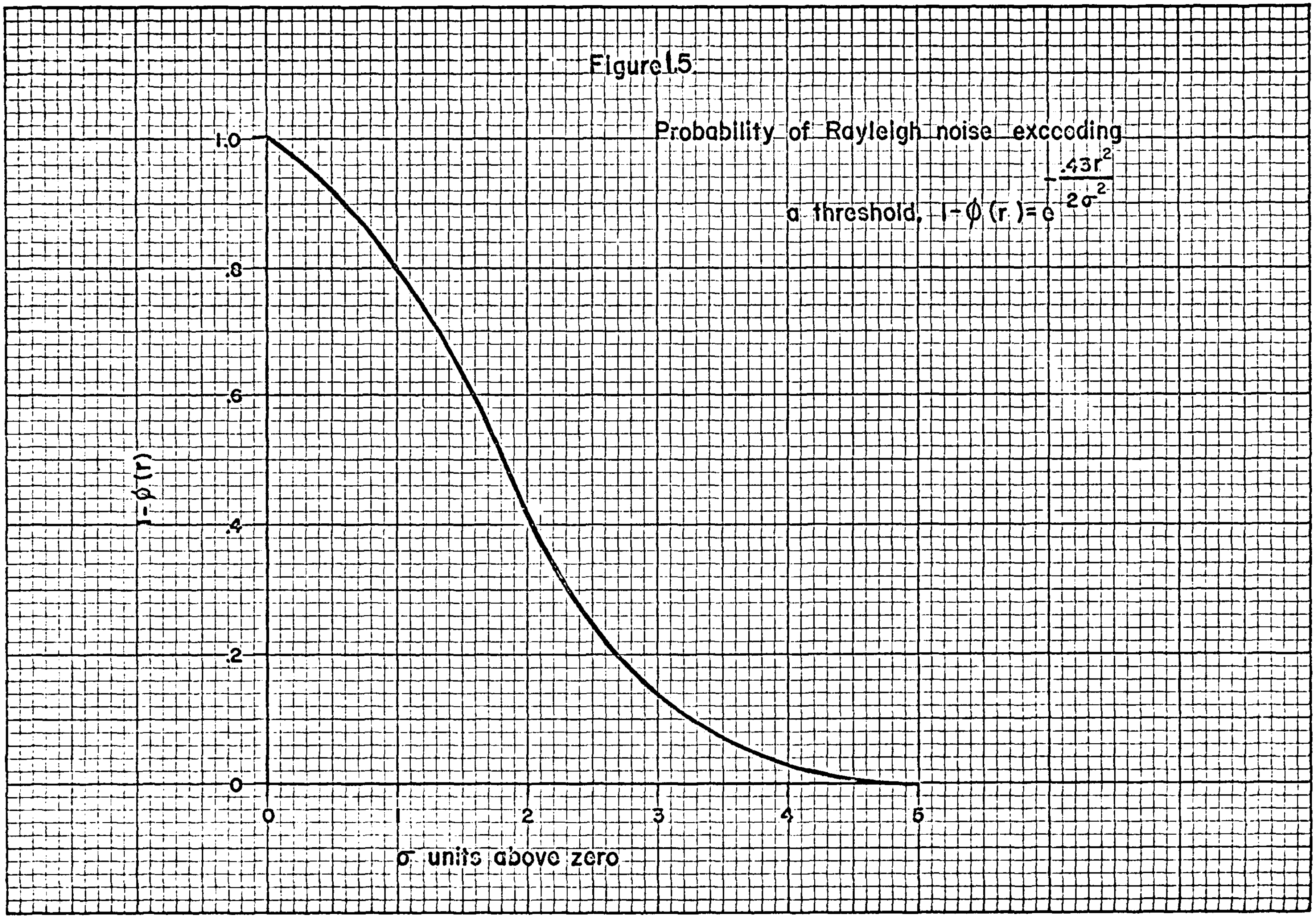


Figure 1.5

Probability of Rayleigh noise exceeding a threshold, $1-\phi(r) = e^{-\frac{.43r^2}{2\sigma^2}}$

obtained using

$$\Gamma\ (1/2) = \sqrt{\pi} \tag{1.44}$$

and the recursion formula

$$\Gamma\ (\alpha + 1) = \alpha\Gamma\ (\alpha) \tag{1.45}$$

Fig.1.6 is a graph of the PDF as a function of χ_n^2 for n = 2, 4, and 10. The distribution function $\phi\ (\chi^2)$ is given by the following formula:

$$\phi\ (\chi^2) = \int_0^{r^2} p\ (\chi^2)\ d\chi^2 \tag{1.46}$$

This function is plotted for n = 2, 4, and 10 in Fig.1.7.

The continuous distributions considered so far are all characterized by one factor that is an exponential and another factor that is some power of the variable. Differences show up in this power to which the variable is raised.

Student T Distribution with n Degrees of Freedom

The Student T probability density function is given by the following formula:

$$p\ (t_n) = \frac{\Gamma\ [(n + 1)/2]}{\sqrt{\pi n}\ \Gamma\left(\frac{n}{2}\right)} \left[1 + \frac{t^2}{n}\right]^{-(n + 1)/2} \tag{1.47}$$

This distribution ranges from minus infinity to plus infinity. The curves are somewhat similar to each other except that for the higher n's; they become more peaked. All have maxima at t = 0. For larger n the Student T distribution

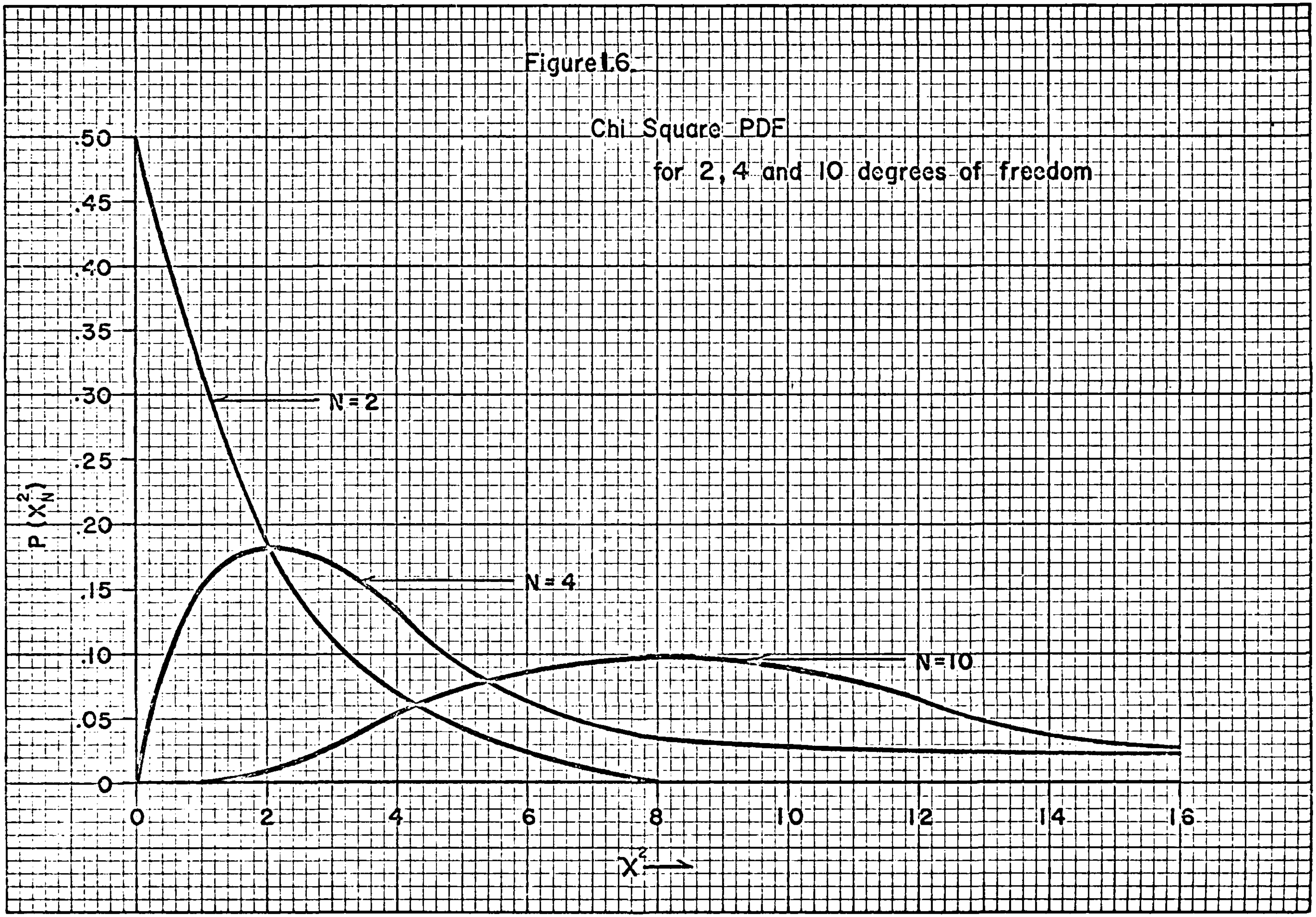
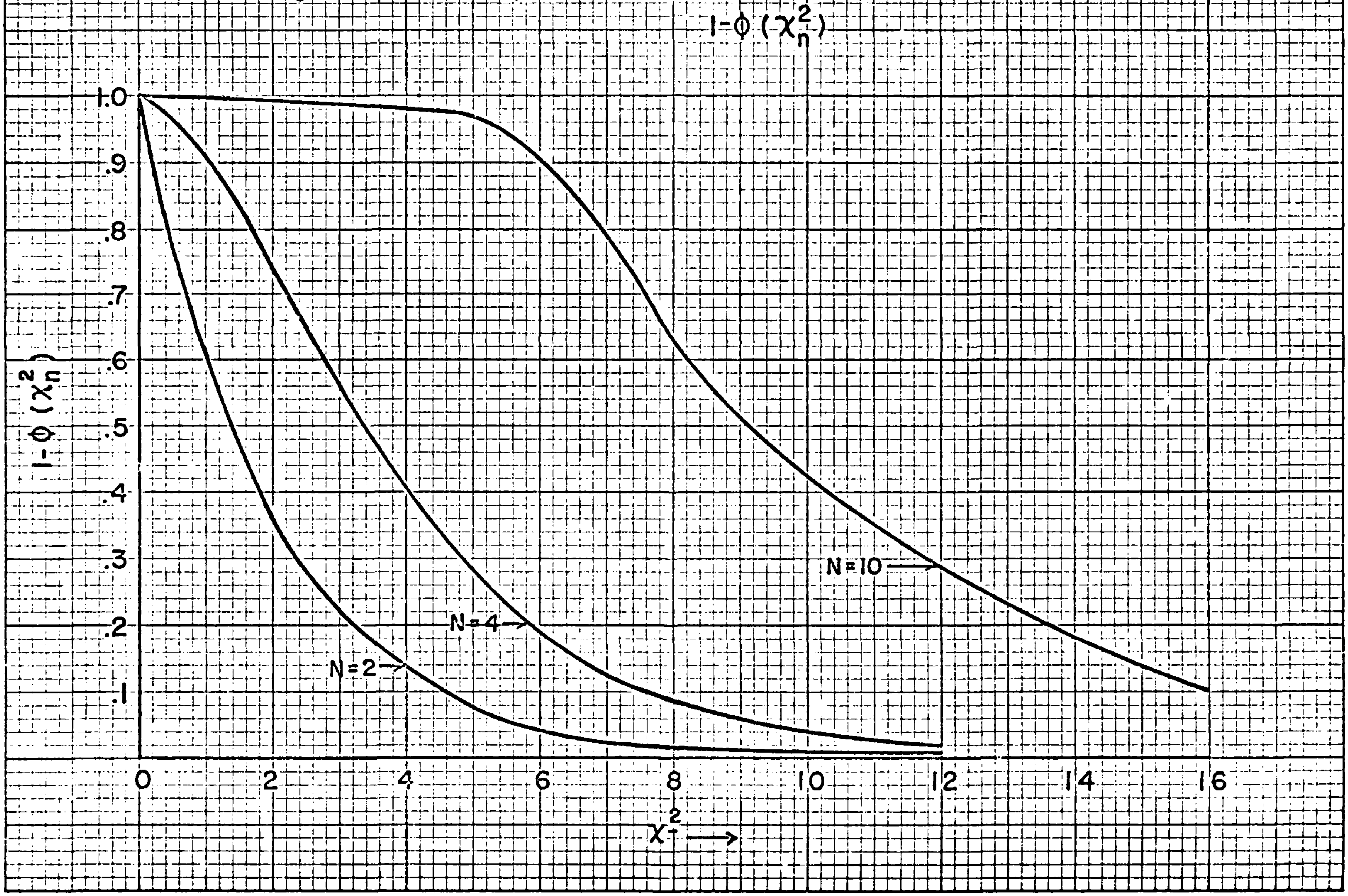


Figure 1.6.

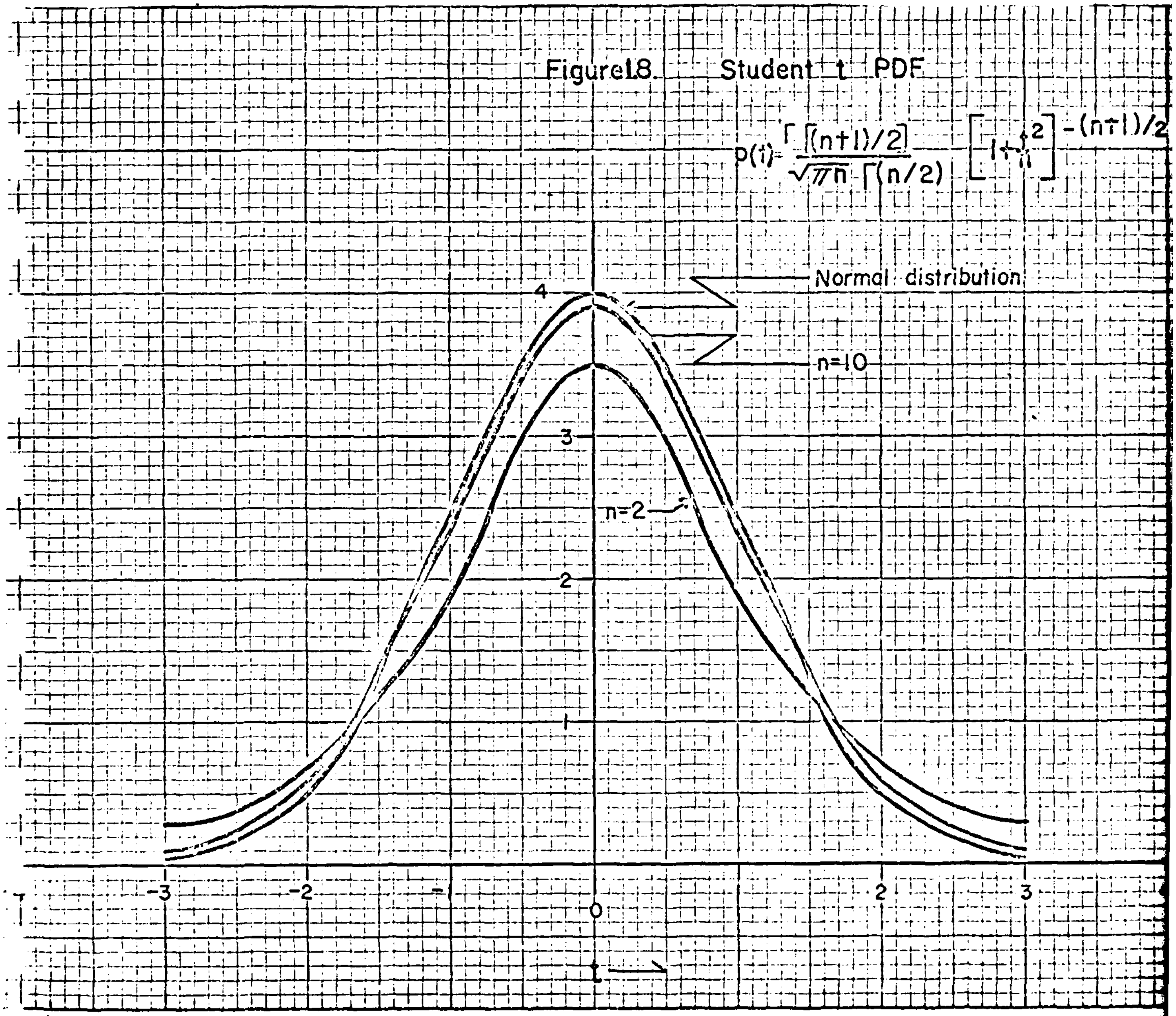
Chi Square PDF for 2, 4 and 10 degrees of freedom

Figure 1.7. Chi Square Distribution



(Student T distribution) approaches normal distribution.

Fig.1.8 shows a graph of Student T PDF for the cases n = 2 and 10 along with the PDF for normal distribution for comparison. Fig.1.9 shows the corresponding curves for the probability of exceeding a value t.



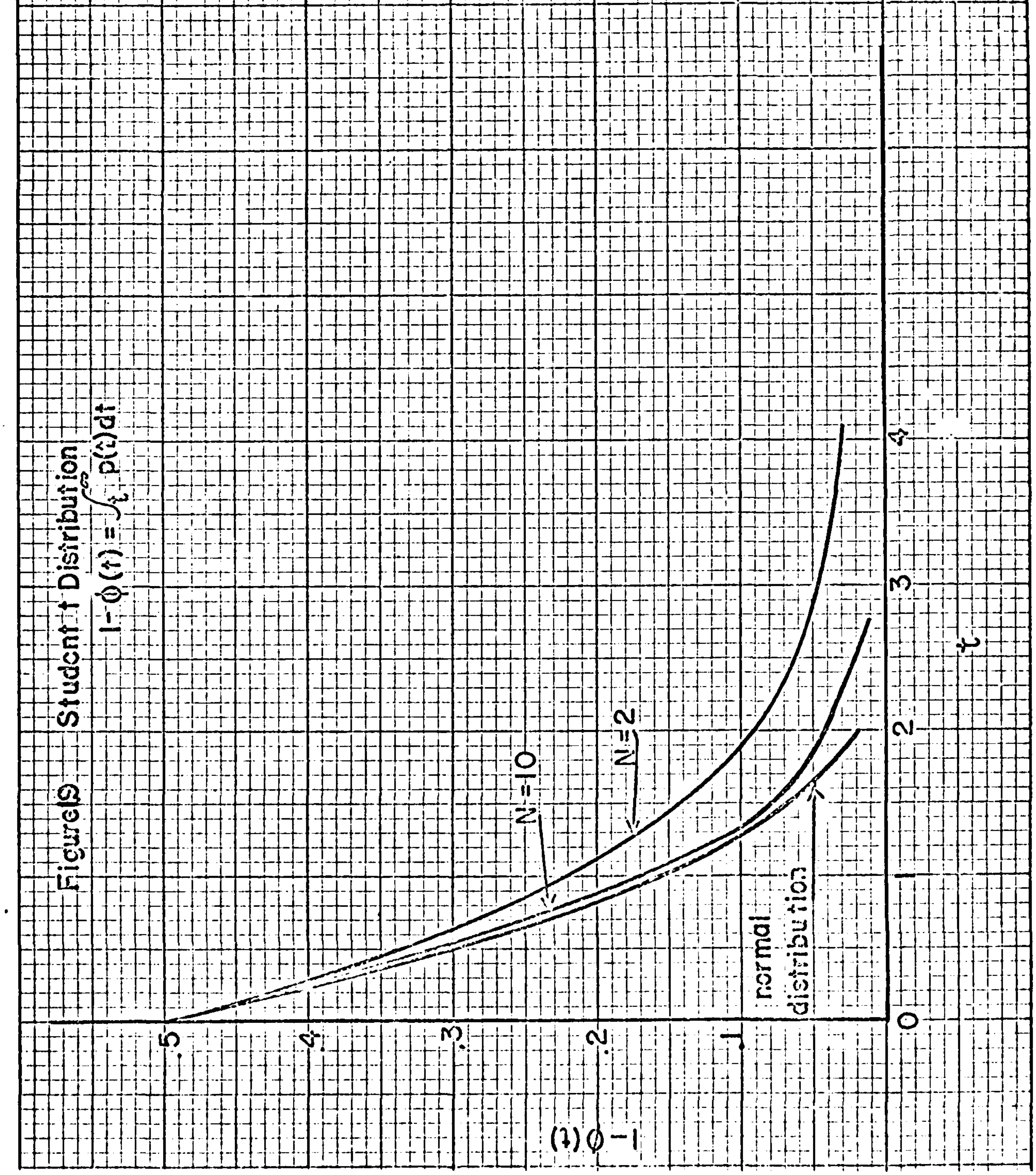


Figure 9 Student t Distribution
$1-\phi(t) = \int_t^{\infty} p(t)dt$
N=10
N=2
normal distribution
.5
.4
.3
.2
.1
$1-\phi(t)$
0
1
2
3
4
t

Joint Probabilities of Independent Events

We can imagine a time series of n random samples, each sample being a 1 or a 0. The 1's represent information calling for a different response or perhaps no response. We let all samples be independent of each other, but we let the probabilities of each sample being a 1 or a 0 be p or q respectively. What is the probability of exactly m 1's being present?

First we consider the probability of m specific 1's and the remaining n - m samples, all 0's. For this case of joint probability, we simply multiply together the probabilities of every sample being what we have specified. There are then m p's and (n - m) q's multiplied together to give

$$P_n(m)_{spec} = p^m q^{n-m}. \tag{1.48}$$

The subscript "spec" means specific samples provide the m 1's.

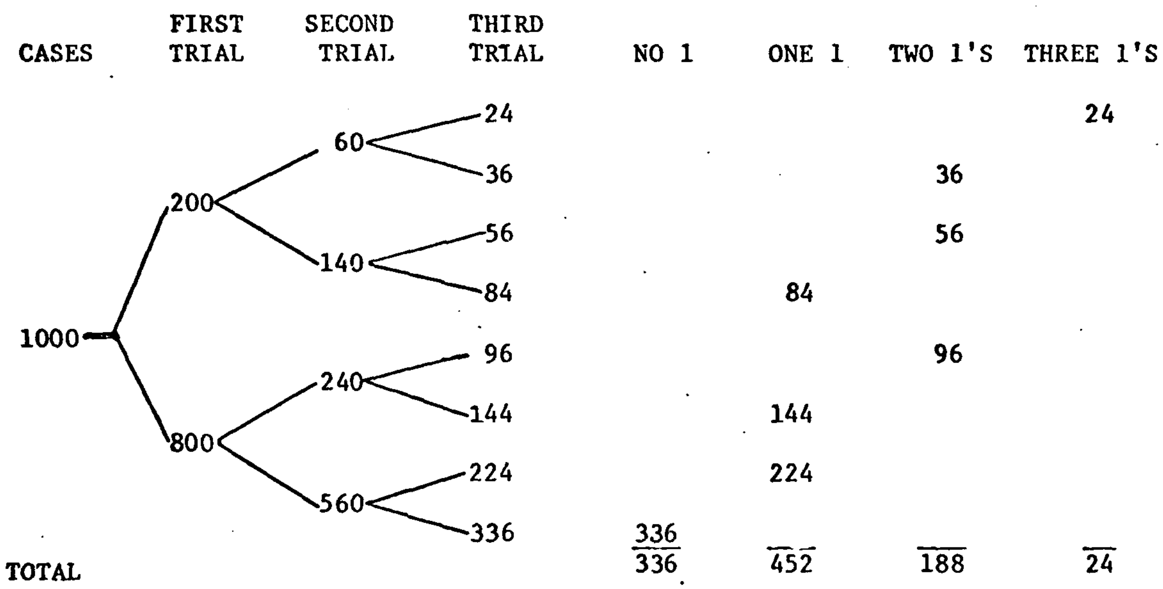
If the question permits any m 1's instead of a specific set, the probability is that for a specific set of m 1's multiplied by the number of specific sets that are possible. As an example, suppose that we are given a probability of 2/10 that any sample is a 1. Let us find the probability that three samples will contain any two 1's. A specific pair of 1's is selected as 1's on the first and second samples with a 0 on the third sample. The probability of this is $.2 \times .2 \times .8 = .032$. But 1's on first and third samples and on second and third samples are also counted so that we have three combinations, each of probability of .032 or a total probability $3 \times .032 = .096$.

The formula for this case is

$$P_n(m) = C_m^n p^m q^{n-m} \tag{1.49}$$

in which C^n_m is the number of combinations of n things taken m at a time. The whole expression is the mth term in the binomial expansion of $(q + p)^n$.

The case method to be explained next will make the picture crystal clear. This method may be profitably resorted to in many statistical problems. We shall introduce some variability here into the probabilities on successive trials p = .2, .3, .4 respectively. Of course q = .8, .7, .6 respectively. Let us consider 1000 cases the number being chosen so that taking some number of tenths three times will always give integers. Then the following flow diagram tells the whole story. Lines slanting up mean 1's, and lines slanting down mean 0's. After each slanting line the number of cases falling in that category is given.



At each trial the assigned probability was applied to all the cases. Let us follow a specific case of a 1 on trials 1 and 3 and a 0 on trial two.

First trial (p = .2): .2 × .1000 = 200 indicated after upward line.
Second trial (q = .7): .7 × 200 = 140 indicated after downward line.
Third trial (p = .4): .4 × 140 = 56 indicated after upward line.

The totals at the bottom of the diagram are sums of all combinations. For two 1's out of three trials we have 188 out of the 1000 cases for a probability of .188.

Returning to Eq. (1.49), we next compute P_5 (m) for five trials (a) with p = .1 and (b) with p = .5 in order to show the gain in ability to distinguish between these cases by establishing a criterion involving multiple trials.

m	$\longrightarrow$	0	1	2	3	4	5
P (m)	(p = .1)	.592	.327	.073	.0081	.0004	----
P (m)	(p = .5)	.031	.145	.313	.313	.156	.031

Suppose now that the criterion for action is that there be three or more 1's out of five. For p = .1, P (n $\geq$ 3) = .0085 and for p = .5, P (m $\geq$ 3) = .5. The ratio of the two values of P (m $\geq$ 3) is about twenty-four times the ratio of the p's. This is a considerable gain in distinguishing between these cases by using a multiple sample criterion rather than a single sample criterion. If the criterion is four out of five, the gain is much greater still, but more cases requiring action are missed.

The mathematics involved here is extensible to any number of mutually exclusive answers, for example, "yes," "no," or "maybe." If the probabilities on a single trial are respectively p, q, and r, and if there are m trials, we

(m trials, we) wish to find the probability of i specific "yeses" as on trials one, six, seven, and nine, and j specific "noes," and m - i - j "maybes." Then the following expression for $(P_m)_{spec}$ applies:

$$(P_m)_{spec} = p^i\, q^j\, r^{m-i-j}. \tag{1.50}$$

An example is the following: given p = 2/10, q = 5/10, and r = 3/10, we ask what is the probability of three "yeses," one "no," and two "maybes," each on specified trials out of six total trials. Substituting, we obtain

$$(P_6)_{spec} = (2/10)^3 \times (5/10)\ (3/10)^2 = 3.6 \times 10^{-4}. \tag{1.51}$$

We consider the basis for Eq. (1.50). Take n sets of six trials each, with n very large. If we take as our specific case trials one and five "yes," trials two, three, and six "no," and trial four "maybe," then out of our n sets there will be np successes on the average on the n trials one. This action reduces the number of cases that can fulfill our requirement to np, the remainder, n (1 - p), having failed already to qualify. Of the np cases, all of which survived the first trial, npq will survive the second. Of these npqq will survive the third and ultimately $npqqrpq = np^2\, q^3\, r$ will qualify, on the average. This number divided by n gives the probability of obtaining the specific result required, and this checks Eq. (1.50).

Next, remove the specification of which particular trials must succeed out of each set and let <u>any</u> i "yeses," j "noes," and m - i - j

("noes," and $m - i - j$) "maybes" out of m trials suffice. Then the preceding result must be multiplied by the number of specific ways in which this result may be obtained. We shall indicate this number by C_{ij}^{m}. The number of combinations of "yeses" that satisfy the requirement is C_i^m, standing for i "yeses" out of m trials. For each of these there are $m - i$ trials per set remaining, and the chance of just j "noes" in these remaining cases is C_j^{m-i} so

$$C_{ij}^{m} = C_i^m \, C_j^{m-i}. \tag{1.52}$$

Let us note that we could start with the "noes" and arrive at

$$C_{ij}^{m} = C_j^m \, C_i^{m-j}.$$

These last two formulas may be checked against each other. We let $m = 4$, $i = 2$, and $j = 1$. The first formula gives $\frac{4 \times 3 \times 2 \times 1}{(2 \times 1)(2 \times 1)} \times \frac{2 \times 1}{1 \times 1} = 12$, and the second formula gives $\frac{4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)\, 1} = \frac{3 \times 2 \times 1}{(2 \times 1)\, 1} = 12$.

The final formula for this case is

$$P_m\,(i, j) = C_i^m \, C_j^{m-i} \, p^i \, q^j \, r^{m-i-j}. \tag{1.53}$$

FOURIER SERIES

We may consider any function of x having only a finite number of discontinuities in a finite range of x in which the function is defined. Consider at first a specific range of x, namely $-\pi$ to $+\pi$. This function is expressible in terms of an infinite sum of sines and cosines with constant coefficients, as follows:

(coefficients, as follows:)

$$f(x)\Big|_{-\pi}^{+\pi} = \sum_{i=0}^{\infty} (a_i \cos ix + b_i \sin ix). \tag{1.54}$$

We proceed at once to deriving the coefficients. Multiply both sides by cos jx and integrate from $-\pi$ to $+\pi$.

The integral on the left is $\int_{-\pi}^{\pi} f(x) \cos jx \, dx$.

On the right there is a sum of integrals of the forms $\int_{-\pi}^{\pi} a_i \cos ix \cos jx \, dx$ and $\int_{-\pi}^{+\pi} a_i \sin ix \cos jx \, dx$. All of these may be shown to be zero except that when $i = j$ and both trigonometric functions are cosines. This result characterizes a set of functions called normal functions. For the last case, there is obtained

$$a_j \int_{-\pi}^{+\pi} \cos^2 jx \, dx = \pi a_j \qquad j \neq 0 \tag{1.55}$$

$$a_0 \int_{-\pi}^{+\pi} dx = 2\pi a_0. \tag{1.56}$$

Thus, equating right and left integrals and dividing by π or 2π, we obtain

$$a_j = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos jx \, dx \qquad j \neq 0 \tag{1.57}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx. \tag{1.58}$$

In like manner, it can be shown that

$$b_j = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin jx \, dx. \tag{1.59}$$

(be shown that)

A simple example follows; we let $f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$.

The work is simplified by noting the f (x) is an odd function. All odd functions are expressible in terms of the sines only. We have

$$b_j = \frac{1}{\pi} \int_{-\pi}^{0} - \sin jx \, dx + \frac{1}{\pi} \int_{0}^{\pi} \sin jx \, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} \sin jx \, dx = \frac{2}{\pi} \left| \frac{-\cos jx}{j} \right|_{0}^{\pi}$$

$$= \frac{2}{\pi} \left(\frac{1 - (-1)^j}{j} \right)$$

$$b_j = 0 \qquad j = 2, 4, 6, 8, \ldots\ldots (j \text{ even})$$

$$b_j = \frac{4}{\pi j} \qquad j = 1, 3, 5, 7, \ldots\ldots (j \text{ odd}).$$

Let us plot the sum of the first three terms to see how closely they approximate the function. This is done in Fig.1.10. The following observations are pertinent. In the flat upper and lower portions of the curve, the maximum deviations from a fit are about 20%, except near x = 0 and near $x = \pm 180^0$, at which values the development gives zero. These exceptions are characteristic of the generality that at a discontinuity the development gives the average value. Another generality is that the development always describes a periodic function with one period traversed in the range of definition, which accounts in our special case for the discontinuities at $x = \pm\pi$ where

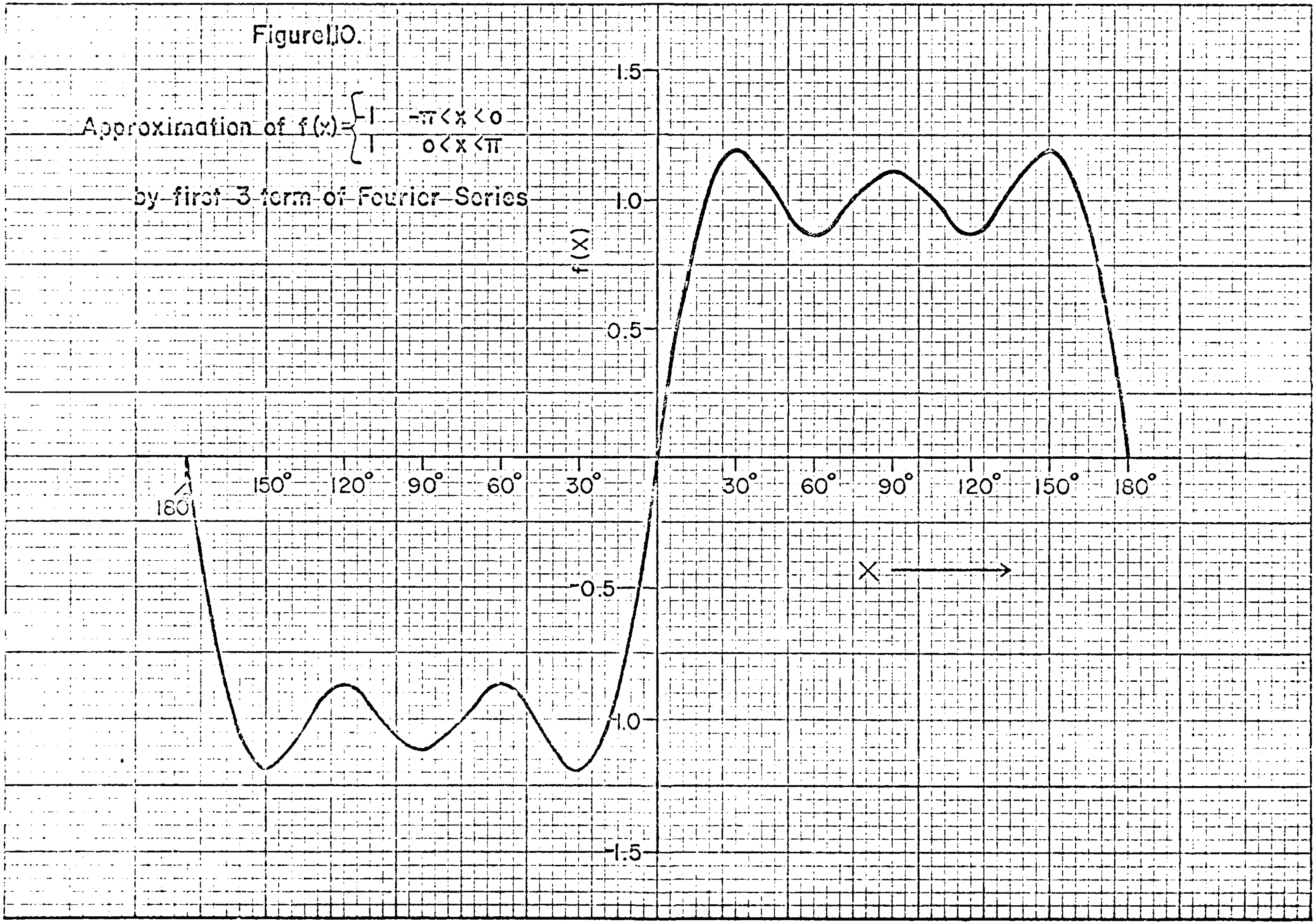


Figure 10.
Approximation of f(x)= { -1 -π<x<0 ; 1 0<x<π }
by first 3 term of Fourier Series
1.5
1.0
0.5
-0.5
-1.0
-1.5
f(X)
180
150°
120°
90°
60°
30°
30°
60°
90°
120°
150°
180°
X

(at $x = \pm\pi$ where) the upper and lower values of the periodic function average zero.

The period 2π from $-\pi$ to $+\pi$ is easily changed to $-\frac{X}{2}$ to $+\frac{X}{2}$ by changing the arguments of the cosines and sines so that when $x = \frac{X}{2}$ the arguments are $i\pi$. This requires that the arguments be $\frac{i2\pi x}{X}$. Eq. (1.59) converts to the following equation:

$$f(x)\Bigg|_{-\frac{X}{2}}^{\frac{X}{2}} = a_0 + \sum_{i=1}^{\infty}\left(a_i \cos\frac{i2\pi x}{X} + b_i \sin\frac{i2\pi x}{X}\right). \qquad (1.60)$$

Let us go through the evaluation of the coefficients as before. We should become very familiar with this step as we may forget the Eqs. (1.57), (1.58), (1.59), and the corresponding formulas to be derived for the present more general case. Multiply both sides by $\cos j\,\frac{2\pi x}{X}$ and integrate the right side term by term from $-\frac{X}{2}$ to $+\frac{X}{2}$. All terms will integrate to zero except

$$a_j \int_{-\frac{X}{2}}^{\frac{X}{2}} \cos^2\left(\frac{j2\pi x}{X}\right) dx = \frac{X\,a_j}{2} \qquad j \neq 0 \qquad (1.61)$$

$$a_0 \int_{-\frac{X}{2}}^{\frac{X}{2}} dx = X\,a_j \qquad j = 0. \qquad (1.62)$$

Equating these to the integral on the left, we obtain

$$a_0 = \frac{1}{X}\int_{-\frac{X}{2}}^{\frac{X}{2}} f(x)\,dx. \qquad (1.63)$$

This is the average value of $f(x)$ in the interval $-\frac{X}{2}$ to $+\frac{X}{2}$. For

(to $+ \frac{X}{2}$. For) $j \neq 0$, we obtain

$$a_j = \frac{2}{X}\int_{-\frac{X}{2}}^{\frac{X}{2}} f(x) \cos jx \, dx \qquad j \neq 0. \quad (1.64)$$

In like manner of derivation, we obtain

$$b_j = \frac{2}{X}\int_{-\frac{X}{2}}^{\frac{X}{2}} f(x) \sin jx \, dx. \qquad (1.65)$$

In applying the general Eq. (1.60), we must first determine whether the function is even, in which case only the a's need be determined. It may be worth mentioning that every analytic function can be expressed as an even function plus an odd function. Indeed, in the general expansion of Eq. (1.52) the constant term and the cosine terms describe the even part and the sine terms describe the odd part. Some functions are odd except for a constant term, in which case only a_0 of the a's needs to be evaluated, e.g. $f(x) = 0 \,\Big|_{-\pi}^{0}$ $f(x) = 1 \,\Big|_{\pi}^{0}$. Since a_0 is always average value in the whole interval of definition, it is obviously 0.5 for the example.

While the Fourier series properly describes a function defined in an interval X, it is, in fact, periodic with period X so that it repeats as x goes through values outside of the interval X.

If we substitute for period X, the reciprocal of frequency, and for x, the variable t, the arguments take the form $i2\pi f_1 t$. In most applications t will be interpreted as time. The set of frequencies if_1 $(i = 0, 1, 2\cdots)$ constitute the line spectrum of the periodic function. If we put

$$a_i \cos(i2\pi f_1 t) + b_i \sin(i2\pi f_1 t) = \sqrt{a_i^2 + b_i^2} \cos(i2\pi f_1 t - \theta)$$

(If we put) with

$$\theta = \tan^{-1} \frac{b_i}{a_i},$$

$\sqrt{a_i^2 + b_i^2}$ appears as the amplitude of frequency if_i and $\tan^{-1} \frac{b_i}{a_i}$ is the epoch angle. It is pointed out here for analogy with results brought out in treating the Fourier integral that even functions have 0° or 180° epoch angle and odd functions have + or - 90° angle.

Fourier Transform of a Transient Function

The particular function used here is chosen because it will later be given a physical interpretation. This function is u (t). For the present, however, consider u as simply a mathematical function of the variable t. The Fourier transform of u will be denoted by U (f) and is given by

$$U\,(f) = \int_{-\infty}^{+\infty} u\,(t)\, e^{-\,i\,2\pi\,ft}\, dt, \tag{1.66}$$

in which $i = \sqrt{-1}$. Since

$$e^{-\,i\,2\pi\,ft} = \cos 2\pi\,ft - i \sin 2\pi\,ft, \tag{1.67}$$

it will be appreciated that when u is an even function, u cos 2π ft is even and $\int_{-\infty}^{+\infty} = 2 \int_0^{\infty}$. On the other hand, u sin 2π ft is then odd and $\int_{-\infty}^{+\infty} = 0$. Thus, for u even we write

$$U\,(f) = 2 \int_0^{\infty} u \cos 2\pi\,ft\, dt, \tag{1.68}$$

and for u odd we write

$$U(f) = -2i \int_0^{\infty} u \sin 2\pi\,ft\, dt. \tag{1.69}$$

(even we write) Very frequently u will be either even or odd, and the transformation using either Eq. (1.68) or Eq. (1.69) becomes only half as difficult as using Eq. (1.67). However, Eq. (1.66) may in some cases be simplest without conversion by Eq. (1.67). Note that the Fourier integral Eq. (1.66) introduces no periodicity as did the Fourier series.

An inverse transform exists for transforming U (f) to u (t). This is

$$u\ (t) = \int_{-\infty}^{+\infty} U\ (f)\ e^{i\ 2\pi\ ft}\ df. \tag{1.70}$$

Similar comments on the function U (f) being even or odd to those made in the preceding paragraph regarding u (t) apply here also. We shall pursue this inverse transform no further except to say that it is a bit tricky to reconstruct a transient from its transform by means of the inverse transform.

Before directing the reader to a reference for very complete treatment, we may remark that frequently $2\pi\ f$ is replaced by ω and df by $\frac{d\omega}{2\pi}$, which renders the forward and inverse transformations less symmetrical. Discussion in terms of ω rather than f frequently meets with favor, but in this text we will favor talking about f, which is frequency in most of the problems with which we shall later deal. Now, for the very complete coverage of Fourier transforms, consult Reference 2.

As an example of the transformation of a transient function of t, let us take $u\ (t) = \sin 2\pi\ f_0\ t\ \Big|_{-\frac{T}{2}}^{\frac{T}{2}}$. Since this is an odd function, Eq. (1.69) is applicable. We write

(applicable. We write)

$$U(f) = -2i \int_0^{\frac{T}{2}} \sin 2\pi f_0 t \sin 2\pi ft \, dt$$

$$= i \int_0^{\frac{T}{2}} \left[\cos 2\pi (f + f_0) t - \cos 2\pi (f - f_0) t\right] dt$$

$$= i \left[\frac{\sin \pi (f + f_0) T}{2\pi (f + f_0)} - \frac{\sin \pi (f - f_0) T}{2\pi (f - f_0)}\right]$$

$$= \frac{iT}{2} \left[\text{sinc}(f + f_0) T - \text{sinc}(f - f_0) T\right].$$

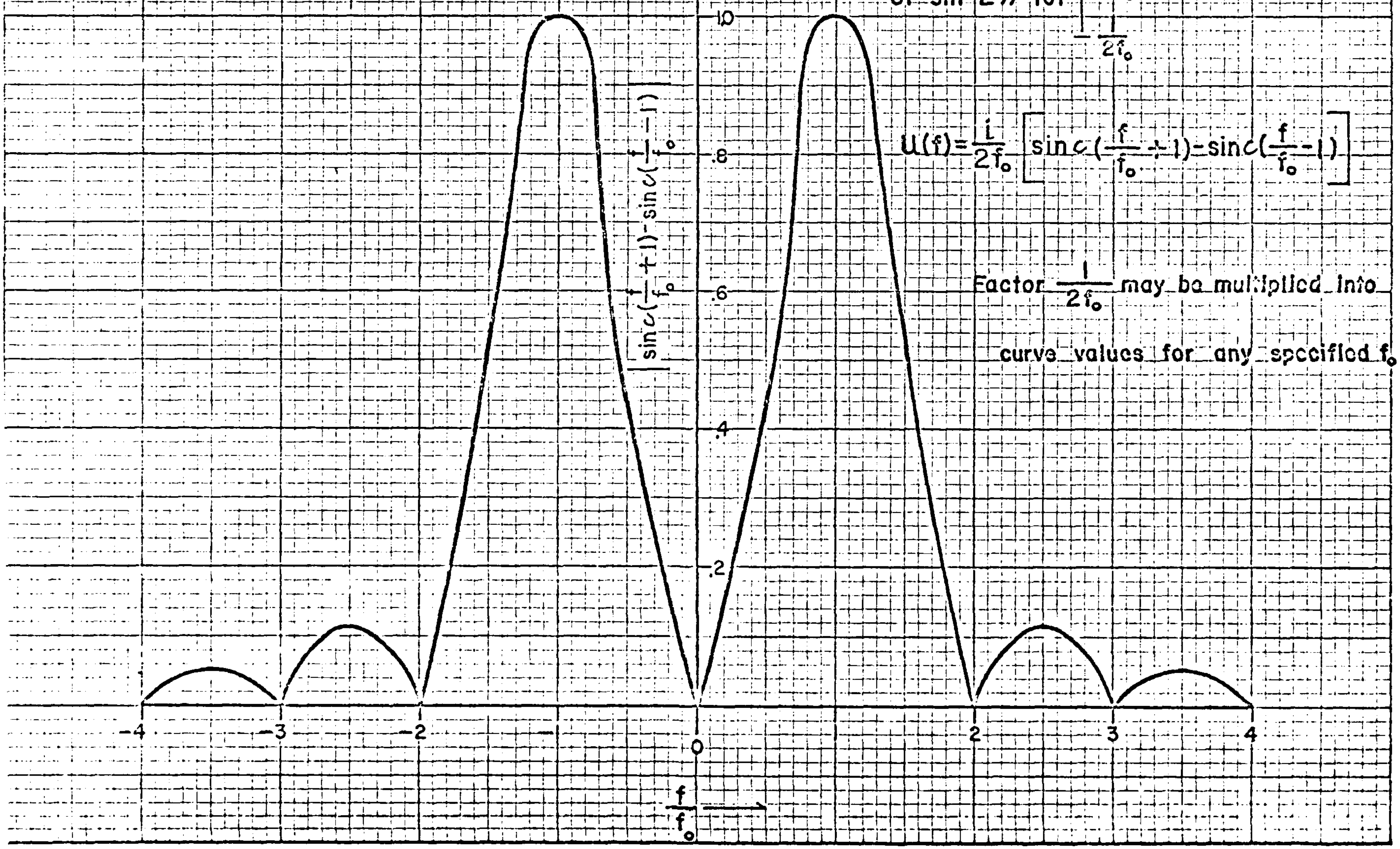
The value U (f) for this case is plotted in Fig.1.11 for $T = \frac{1}{f_0}$ for which u (t) is a single cycle. The significance of the i becomes apparent when we associate t with time and f with frequency. The function U (f) contains a component in any interval f, which is an odd harmonic function of time (not expressed in the frequency expansion). If we had transformed $\cos 2\pi f_0 t$, there would have been no i involved, and this would have indicated every component an even function of time. In other words, the i designates phase shift from the phase of the cosine term. This phase indicator could be complex for some cases, for example, for the transform of $\sin (2\pi f_0 t - \theta)$, which is neither wholly even nor wholly odd.

We speak of U (f) as a spectral density, but in reality, the absolute value of its square U (f)U *(f) is an energy spectral density, e.g., the energy per Hertz. We shall now solve for this as follows:

$$U(f)\, U^*(f) = \frac{T^2}{4} \left[\text{sinc}(f + f_0) T - \text{sinc}(f - f_0) T\right]^2. \qquad (1.71)$$

As a check on our statements, we are able to integrate over the frequency

Figure 1.11. Absolute value of Amplitude Spectrum of $\sin 2\pi f_0 t \Big|_{-\frac{1}{2f_0}}^{+\frac{1}{2f_0}}$



(over the frequency) band from $-\infty$ to $+\infty$ and obtain

$$\int_{-\infty}^{\infty} U\ (f)\ U*\ (f)\ df = \frac{T}{4}\int_{-\infty}^{\infty}\left[\operatorname{sinc}(f+f_0)\ T - \operatorname{sinc}(f-f_0)\ T\right]^2 T\ df \qquad (1.72)$$

in which the T taken under the integral forms T df, the exact differential of the argument of either sinc function. The integral of each squared sinc is one, and that of the cross product is zero since displaced sinc functions are orthogonal. Thus we have

$$\int_{-\infty}^{+\infty} U\ (f)\ U*\ (f)\ df = \frac{T}{2} \qquad (1.73)$$

which is half the amplitude squared times the duration, or average power times time, or total energy, thereby confirming the statement that U (f) U * (f) is energy spectrum.

CORRELATION FUNCTIONS

Before addressing ourselves to the correlation functions, we shall outline our objectives in this treatment. First, the cross correlation and autocorrelation functions will be defined. Then it will be shown that the average power may be obtained from the autocorrelation function. The Fourier transform of the autocorrelation function will then be shown by an example to be the power spectrum that is converted to energy spectrum previously derived by multiplying by duration T.

For any continuous function u (t), the autocorrelation function is defined by

$$\psi_{11}\ (\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u\ (t)\ u\ (t-\tau)\ dt. \qquad (1.74)$$

(is defined by) For two continuous functions $u(t_1)$ and $u_2(t_1)$, the cross correlation is given by

$$\psi_{12}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u_1(t)\, u_2(t-\tau)\, dt. \qquad (1.75)$$

For transient functions these equations are most meaningful if the limit $T\to\infty$ is not taken. The averaging time should equal maximum overlap in time, which is the duration of the shorter of the two functions. Integration need be carried out for any τ only to include all t for which the integrand $\neq 0$. This is sometimes less than the averaging time.

Putting $\tau = 0$ in Eq. (1.74) obviously gives average power in a one ohm resistor. We say $(\psi_{11})_0$ = average power.

We next consider the autocorrelation function of $\sin 2\pi f_0 t$ between $-\frac{T}{2}$ and $+\frac{T}{2}$, 0 elsewhere. Then the product $\sin 2\pi f_0 t \cdot \sin 2\pi f_0 (t-\tau)$ is 0 except when t lies between $-\frac{T}{2}+\tau$ and $\frac{T}{2}$ for positive τ and between $-\frac{T}{2}$ and $\frac{T}{2}+\tau$ for negative τ. The autocorrelation function then takes the form

$$\left.\begin{aligned} \psi_{11} &= \frac{1}{T}\int_{-\frac{T}{2}+\tau}^{\frac{T}{2}} \sin 2\pi f_0 t \cdot \sin 2\pi f_0 (t-\tau)\, dt \\ &\text{for } \tau \text{ plus and} \\ \psi_{11} &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}+\tau} \sin 2\pi f_0 t \sin 2\pi f_0 (t-\tau)\, dt \end{aligned}\right\} (1.76)$$

for τ negative.

The same integration is involved in both expressions, namely $\int \sin 2\pi f_0 t \sin 2\pi f_0 (t-\tau)\, dt$. The quantity sin · sin equals

(sin • sin equals) cos difference Δ minus cos sum Σ divided by 2. The arguments Σ and Δ are $2\pi f_0 (2t - \tau)$ and $2\pi f_0 \tau$ respectively, so

$$\left.\begin{aligned} \psi_{11} &= \frac{1}{2T}\int_{t_1}^{t_2} \left[\cos 2\pi f_0 \tau - \cos 2\pi f_0 (2t - \tau)\right] dt \\ &= \frac{1}{2T} \left| \tau \cos 2\pi f_0 \tau - \frac{1}{4\pi f_0} \sin 2\pi f_0 (2t - \tau) \right|_{t_1}^{t_2} \end{aligned}\right\} . \quad (1.77)$$

The following equations are obtained by using appropriate limits:

$$\left.\begin{aligned} \psi_{11} &= \frac{1}{2T}\left\{\left[\cos 2\pi f_0 \tau\right] (T - \tau) - \frac{\sin 2\pi f_0 (T - \tau)}{2\pi f_0}\right\}, \tau > 0 \\ &\text{and} \\ \psi_{11} &= \frac{1}{2T}\left\{\left[\cos 2\pi f_0 \tau\right] (T + \tau) - \frac{\sin 2\pi f_0 (T + \tau)}{2\pi f_0}\right\}, \tau < 0 \end{aligned}\right\} . \quad (1.78)$$

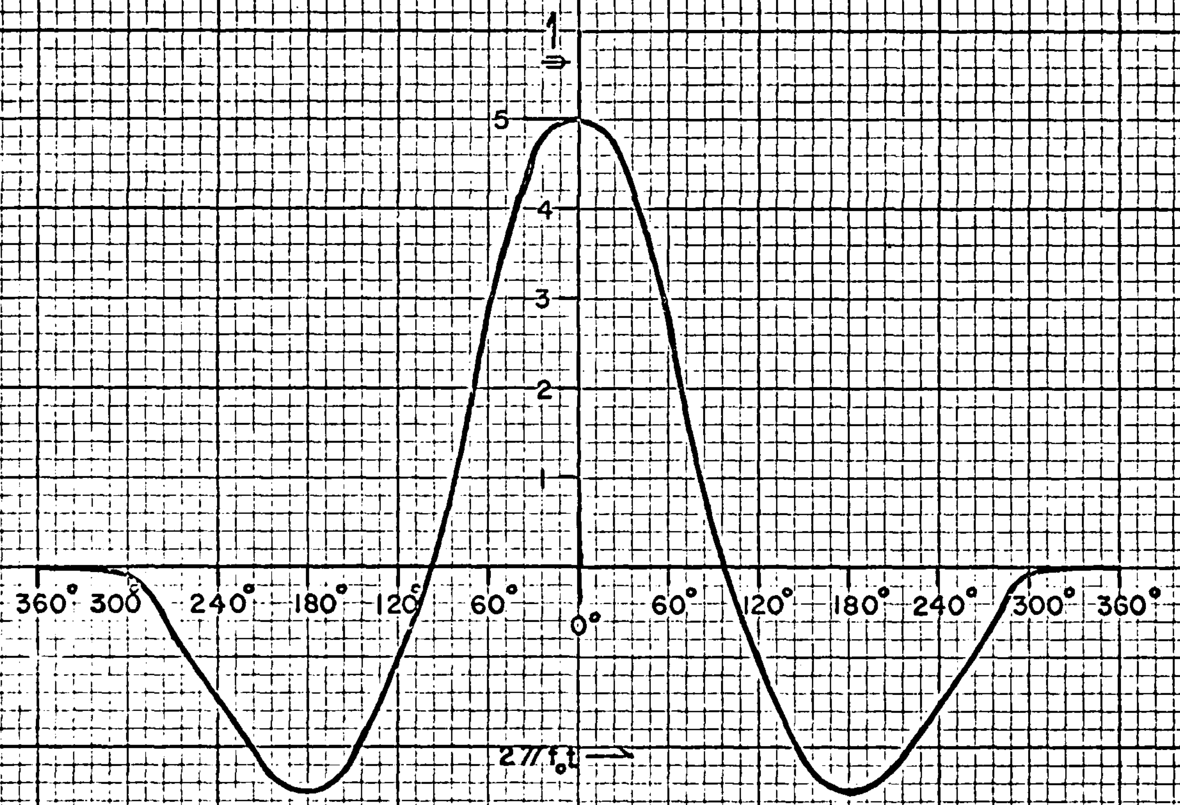
We note that ψ here is an even function since a negative τ in the second expression (that for negative τ) gives the same values as the first expression with τ positive. Moreover, ψ_{11} is <u>always</u> an even function.

Another characteristic of the autocorrelation function is that it goes through a maximum for $\tau = 0$. This may not be apparent in Eq. (1.78), but when the slope is determined by differentiating with respect to τ, it is found that this slope is positive for slightly negative τ, zero for zero τ, and negative for slightly positive τ, with two terms in the derivative nicely cancelling each other at $\tau = 0$. The maximum at $\tau = 0$ is never exceeded by any other maximum. The function ψ_{11} for $u = \sin 2\pi f_0 t$ is plotted versus τ in Fig. 1.12 for $T = \frac{1}{f_0}$, a single cycle.

Figure 112.

Auto correction function of

$\sin 2\pi f_0 t, -\frac{1}{2f_0} \ll t \ll +\frac{1}{2f_0}$



(a single cycle.)

The only comments on cross correlation that will be made here are that usually the two functions being correlated are signal only and signal plus noise and that this is an effective processing method when the signal is spread over a medium to wide bandwidth. For this case a filter would have to accommodate the whole bandwidth and would admit the noise in this relatively wide band. However, cross correlation is equivalent to predetector filtering in a bandwidth equal to the reciprocal of the pulse length regardless of signal bandwidth.

The Fourier transform of the autocorrelation function will now be computed for the case in which the time series is $u = \sin 2\pi f_0 t$, and its correlation function is given by Eq. (1.78), with τ as the time variable. Since the whole correlation function is even, we may replace $e^{-i 2\pi f \tau}$ by $\cos 2\pi f \tau$, multiply all terms of the autocorrelation function by this quantity, and take twice the integral with respect to τ from zero to T since the overlap of the two functions disappears at just $\tau = T$.

We let the Fourier transform be denoted by g. This consists of two parts:

$$a = \frac{2}{T} \int_0^T (\cos 2\pi f_0 \tau) \left(\frac{T}{2} - \frac{\tau}{2}\right) \cos 2\pi f \tau \, d\tau \qquad (1.79)$$

and

$$b = - \frac{1}{2\pi f_0 T} \int_0^T \sin 2\pi f_0 (T - \tau) \cos 2\pi f \tau \, d\tau. \qquad (1.80)$$

Starting with a, we express the product of the cosines as the cosine of the difference plus the cosine of the sum over 2, giving

(over 2, giving)

$$a = \frac{1}{T} \int_0^T \left(\frac{T}{2} - \frac{\tau}{2}\right) \left[\cos 2\pi (f - f_0) \tau + \cos 2\pi (f + f_0) \tau\right] d \tau. \quad (1.81)$$

We integrate by parts

$$u = \frac{1}{2} - \frac{\tau}{2T} \qquad dv = \left[\cos 2\pi (f - f_0) \tau + \cos 2\pi (f + f_0) \tau\right] d \tau$$

$$du = - \frac{-d\tau}{2T} \qquad v = \left[\frac{\sin 2\pi (f - f_0) \tau}{2\pi (f - f_0)} + \frac{\sin 2\pi (f + f_0) \tau}{2\pi (f + f_0)}\right]$$

$$uv\Big|_0^T = 0 \qquad - \int vdu = \frac{1}{2T} \int_0^T \left[\frac{\sin 2\pi (f - f_0) \tau}{2\pi (f - f_0)} + \frac{\sin 2\pi (f + f_0)}{2\pi (f + f_0)}\right] d \tau$$

$$a = - \frac{1}{2T} \left[\frac{\cos 2\pi (f + f_0) \tau}{4\pi^2 (f + f_0)^2} + \frac{\cos 2\pi (f - f_0) \tau}{4\pi^2 (f - f_0)^2}\right]_0^T$$

$$= + \frac{1}{2T} \left[\frac{1 - \cos 2\pi (f + f_0) T}{4\pi^2 (f + f_0)^2} + \frac{1 - \cos 2\pi (f - f_0) T}{4\pi^2 (f - f_0)^2}\right]$$

$$= \frac{T}{4} \left[\frac{\sin^2 \pi (f + f_0) T}{\pi^2 (f + f_0)^2 T^2} + \frac{\sin^2 \pi (f - f_0) T}{\pi^2 (f - f_0)^2 T^2}\right]$$

$$= \frac{T}{4} \left[\sin c^2 (f + f_0) T + \sin c^2 (f - f_0) T\right]. \quad (1.82)$$

Continuing with the term b, Eq. (1.80), we express the product of the sine and the cosine as sine of the sum plus sine of the difference over 2, giving

$$b = - \frac{1}{4\pi f_0 T} \int_0^T \left\{\sin 2\pi \left[f_0 T + (f - f_0) \tau\right] + \sin 2\pi \left[f_0 T - (f + f_0) \tau\right]\right\} d \tau$$

(over 2, giving)

$$b = \frac{1}{4\pi f_0 T} \int_0^T \left\{ \sin 2\pi \left[f_0 T + (f - f_0) \tau \right. + \sin 2\pi \left. f_0 T - (f + f_0) \tau \right] \right\} d\tau$$

$$= \frac{1}{4\pi f_0 T} \left| \frac{\cos 2\pi \left[f_0 T + (f - f_0) \tau \right]}{2\pi (f - f_0)} - \frac{\cos 2\pi \left[f_0 T - (f + f_0) \tau \right]}{2\pi (f + f_0)} \right|^T$$

$$= \frac{\cos 2\pi f T - \cos 2\pi f_0 T}{4\pi^2 (f^2 - f_0^2) T}. \qquad (1.83)$$

Letting $f = x + y$ and $f_0 = x - y$ and substituting, we have

$\cos 2\pi f T = \cos 2\pi T (x + y) = \cos 2\pi x T \cos 2\pi y T - \sin 2\pi x T \sin 2\pi y T$

and

$- \cos 2\pi f_0 T = \cos 2\pi T (x - y) = - \cos 2\pi x T \cos 2\pi y T - \sin 2\pi x T \sin 2\pi y T.$

Adding and replacing x by $\frac{f + f_0}{2}$ and y by $\frac{f - f_0}{2}$, we are left with the product of the sines only, and b becomes

$$b = \frac{- \sin \pi (f + f_0) T \sin \pi (f - f_0) T}{2\pi^2 (f^2 - f_0^2) T}$$

$$= - \frac{T}{2} \operatorname{sinc}(f + f_0) T \operatorname{sinc}(f - f_0) T. \qquad (1.84)$$

Adding Eq. (1.82) and Eq. (1.84), we obtain the perfect square

$$g (f) = - \frac{T}{4} \left[\operatorname{sinc}(f - f_0) T - \operatorname{sinc}(f + f_0) T \right]^2. \qquad (1.85)$$

This differs from U (f) U * (f) given by Eq. (1.71) in that

(Eq. (1.71) in that)

$$g(f) = \frac{1}{T} U(f) U^*(f). \tag{1.86}$$

It is therefore the power spectrum of $u(t)$.

By far the easiest way to obtain $g(f)$ is by way of $u(t)$ and Eq. (1.76). However, the relationship between autocorrelation function and power spectrum is quite useful for general manipulation.

In the course of this development we have obtained average power by squaring the time series, integrating over its duration, and dividing by its duration. We have also seen that the power spectrum can be integrated from $-\infty$ to $+\infty$ to give average power. A statement of the equality of these two expressions for average power is Parseval's theorem.

It has been brought out that no other maximum in an autocorrelation function is greater than the one at $\tau = 0$. This maximum may be any finite value. For many purposes, as for example when comparing two autocorrelation functions, it is desirable to normalize by dividing by a quantity that will make this central maximum unity. This quantity is, of course, the function itself evaluated at $\tau = 0$.

Cross correlation functions generally need to be normalized to bring out a part of their significance. This is done by dividing $\psi_{12}(\tau)$ by $\sqrt{\psi_{11}(0)\,\psi_{22}(0)}$ with ψ_{11} and ψ_{22} not normalized. When this is done, the normalized value will never exceed unity. Its maximum value is a measure of the coherence of the two time functions, unity meaning perfect coherence or, apart from a constant factor, identity.

GAMMA FUNCTIONS.

The Γ function is defined for positive real ν by:

$$\Gamma(\nu) = \int_0^\infty e^{-t}\, t^{\nu-1} dt. \tag{1.87}$$

When ν is an integer n, successive integrations by parts reduce Eq. (1.87) to

$$\Gamma(n) = (n - 1)! \tag{1.88}$$

and as with the factorial we have a recursion formula

$$\Gamma(n + 1) = n\Gamma(n). \tag{1.89}$$

Equation (1.89) holds for any n, real or complex, and may be used to give either ascending or descending arguments (e.g. solving for $\Gamma(-1/3)$ from $\Gamma(2/3)$).

The Eq. (1.87) may be integrated for $\nu = 1/2$ by the substitution $t = u^2$, yielding

$$\Gamma(1/2) = \sqrt{\pi}. \tag{1.90}$$

Then successive applications of Eq. (1.89) yield $\Gamma(3/2)$, $\Gamma(5/2)$, ... , as well as $\Gamma(-1/2)$, $\Gamma(-3/2)$, ...

We record also:

$$\begin{aligned} \Gamma(2/3) &= 1.35412 \\ \Gamma(1/3) &= 2.67893. \end{aligned} \tag{1.91}$$

BESSEL FUNCTIONS

Bessel Functions of the First Kind

In problems involving cylindrical coordinates (e.g. sound propagation in stratisfied media), we encounter Bessel functions. Most useful functions are solutions to differential equations. The Bessel functions may be taken up as solutions of Bessel's equation.

Bessel's equation is

$$\frac{d^2Z}{dz^2} + \frac{1}{z}\frac{dZ}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) Z = 0 \qquad (1.92)$$

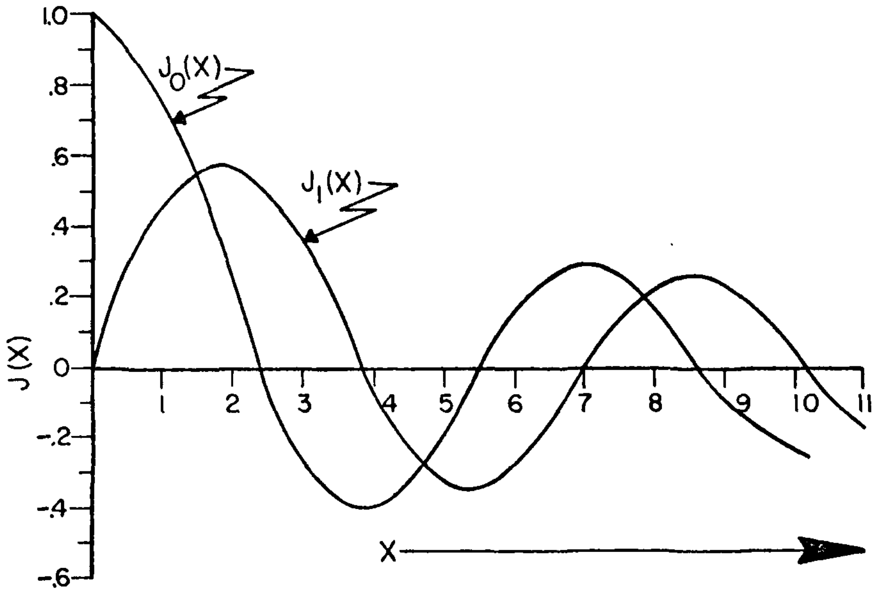
and a solution is $Z = J_\nu(z)$. By writing Z as a power series, substituting in Eq. (1.92) and equating coefficients of the different powers of z in the equation independently to 0, we find certain relationships among the coefficients of the power series for Z that are satisfied by the following formula when ν equals a positive integer n.

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \, (z/2)^{(n + 2m)}}{m! \, (n + m)!}. \qquad (1.93)$$

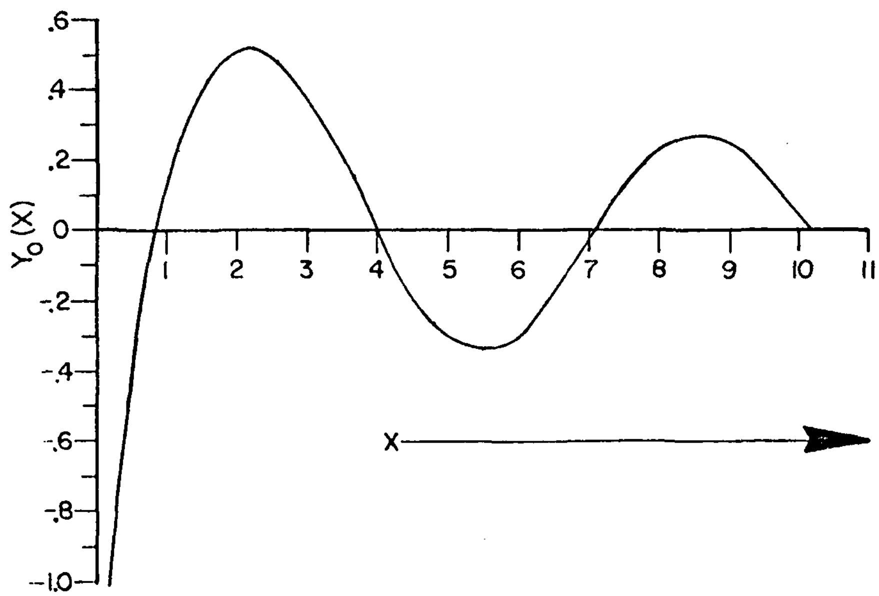
The value $J_n(z)$ is called a Bessel function of the first kind of order n and argument z. The values of $J_o(x)$ and $J_1(x)$ are plotted in Fig.1.13.

For negative integral orders it can be shown that

$$J_{-n}(z) = (-1)^n \, J_n(z). \qquad (1.94)$$



BESSEL FUNCTION OF FIRST KIND



BESSEL FUNCTION OF SECOND KIND

FIGURE 1.13

Equation (1.93) may be expressed in terms of Γ functions as follows:

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \, (z/2)^{(n + 2m)}}{\Gamma(m + 1)\Gamma(n + m + 1)}. \qquad (1.95)$$

It develops that for any positive ν we may substitute this ν for n in Eq. (1.96) obtaining

$$J\ (z) = \sum_{m=0}^{\infty} \frac{(-1)^m \, (z/2)^{(\nu + 2m)}}{\Gamma(m + 1)\Gamma(\nu + m + 1)}. \qquad (1.96)$$

<u>Bessel Functions of the Second Kind</u>

The solution $J_\nu(z)$ evolves from the straight forward procedure just indicated. The student may substitute the solution in the form given in Eq. (1.93) into Eq. (1.92) thereby verifying that Eq. (1.93) is a solution of Eq. (1.92). However, there must be *a second independent solution of a* second order differential equation. This will be given without derivation.

Because ν enters Eq. (1.92) as ν^2, substituting $-\nu$ for ν gives the identical equation. Therefore, $J_{-\nu}(z)$ is a solution of Eq. (1.92). When ν is an integer, $J_{-n}(z)$ and $J_n(z)$ are not independent as can be seen from Eq. (1.94). However, when ν is not an integer, it turns out that $J_{-\nu}(z)$ is linearly independent of $J_\nu(z)$, and a solution

$$Z = AJ_\nu(z) + BJ_{-\nu}(z) \qquad (1.97)$$

constitutes a "fundamental system" of solutions.

It should also be apparent that the combination of $J_\nu(z)$ and a function of $Y_\nu(z)$ which is a linear combination of $J_\nu(z)$ and $J_{-\nu}(z)$ constitutes a fundamental system of solutions since

$$\begin{aligned} aJ_\nu(z) + bY_\nu(z) &= aJ_\nu(z) + b\left[cJ_\nu(z) + dJ_{-\nu}(z)\right] \\ &= (a + bc)\, J_\nu(z) + bdJ_{-\nu}(z) \\ &= AJ_\nu(z) + BJ_{-\nu}(z) \end{aligned}$$

A precise relationship between $Y_\nu(z)$ and the J's is commonly used, namely

$$Y_\nu(z) = \frac{J_\nu(z) \cos \nu\pi - J_{-\nu}(z)}{\sin \nu\pi} \tag{1.98}$$

The Eq. (1.98) defines a Bessel function of the second kind and is called a Weber function. (Ref. 3)

When ν is an integer n, $Y_n(z)$ is indeterminate from Eq. (1.98). However, it may be defined as the limit of the right side of Eq. (1.98) as ν converges to n. It has been evaluated and is linearly independent of $J_n(z)$; therefore, we have the equation

$$Z = AJ_\nu(z) + BY_\nu(z) \tag{1.99}$$

which holds for all real ν. $Y_0(X)$ is plotted in Fig. 1.13. $Y_0(0) = -\infty$.

By using kz in place of z in Eq. (1.92) and taking the special case of $\nu = 0$, we obtain

$$\frac{d^2Z}{dz^2} + \frac{1}{z}\frac{dZ}{dz} + k^2Z = 0 \tag{1.100}$$

a form which may develop in the process of solving the wave equation in cylindrical coordinates with k^2 introduced in the separation of variables. Since we have substituted kz for z, a solution is $J_o(kz)$. A fundamental system of solutions is

$$Z = AJ_o(kz) + BY_o(kz). \tag{1.101}$$

It sometimes develops that only certain values of k permit satisfying boundary conditions. These permitted values of k are called eigenvalues, and the Z obtained for any eigenvalue is an eigenfunction belonging to its eigenvalue, which in turn belongs to it. Boundary conditions may also require that either A or B be zero.

Bessel Functions of the Third Kind

There are two functions of the third kind, usually called Hankel functions, and designated $H_\nu{}^1$ and $H_\nu{}^2$. These functions are given by

$$H_\nu{}^1(z) = J_\nu(z) + iY_\nu(z) \tag{1.102}$$

and

$$H_\nu{}^2(z) = J_\nu(z) - iY_\nu(z). \tag{1.103}$$

Either Hankel function is a solution of Eq. (1.92), and the two together comprise a fundamental system of solutions. In some solutions to the wave equations we shall use this fundamental system of solutions. Specifically, we shall encounter a solution

$$Z = EH_o{}^1(z) + FH_o{}^2(z) \tag{1.104}$$

where E and F are constants. The function $H_o^1(z)$ will be interpreted as an incoming* wave and H_o^2 as an outgoing* wave so that the two together may comprise standing waves and in addition traveling waves (when $E \neq F$).

Since J and Y are real, the Hankel functions are complex and conjugate to each other. The interpretation of a complex solution is an amplitude $|H_o^1|$ or $|H_o^2|$, and a phase lead $\tan^{-1}(\pm Y/J)$, the plus sign for H_o^1, and minus sign for H_o^2. The amplitude and phase are tabulated in Ref. 9.

We should note that this complex form is analogous to our use of $e^{\pm ikz}$ to stand for harmonic space functions. As a matter of fact, the limiting forms of J_o, Y_o, and H_o as the argument increases without limit are cosines, sines, and exponentials as follows:

$$\lim_{z\to\infty} J_o(z) = (2/\pi z)^{1/2} \cos\left(z - \frac{\pi}{4}\right), \tag{1.105}$$

$$\lim_{z\to\infty} Y_o(z) = (2/\pi z)^{1/2} \sin\left(z - \frac{\pi}{4}\right), \tag{1.106}$$

and

$$\lim_{z\to\infty} H_o^{1,2}(z) = (2/\pi z)^{1/2} e^{\pm i}\left(z - \frac{\pi}{4}\right). \tag{1.107}$$

The phase for each of these functions approaches $(z - \frac{\pi}{4})$ at large z. The major difference from the harmonic functions is the factor $z^{-1/2}$, which will be associated in our applications with divergence loss characteristic of cylindrical spreading.

---

* The respective roles of H_o^1 and H_o^2 are often reversed arbitrarily to match an arbitrary choice of the negative sign in $e^{+i\omega t}$ in the solutions of the wave equation which will follow. We shall use $e^{+i\omega t}$ and this convention determines the specified interpretations of H_o^1 and H_o^2.

Modified Bessel Functions*

In physical problems we sometimes encounter negative and complex wave numbers. As an example, if we have the complex wave number

$$k = k_o + ik_1 \qquad (1.108)$$

it follows that

$$e^{ikx} = e^{ik_0x} e^{-k_1x}. \qquad (1.109)$$

The last factor is an exponential damping factor. A factor e^{-k_1z} in the Hankel functions of zero order with large arguments would emerge from Eq. (1.107) if z were replaced by kz, k given by Eq. (1.108).

Imaginary wave numbers will often occur in problems of stratified media. These may appear in the arguments of a Bessel function in such a way as to render the argument imaginary. Bessel functions of the three kinds are all given special designations for imaginary arguments and are all called "modified" Bessel functions. We have the relationships

$$I_\nu(z) = J_\nu(iz) \qquad (1.110)$$

$$K_\nu(z) = Y_\nu(iz) \qquad (1.111)$$

$$L_\nu^{1,2}(z) = H_\nu^{1,2}(iz). \qquad (1.112)$$

It should be noted that imaginary arguments lead to modified functions which are real. Furthermore, the variable z here is not necessarily depth.

* The meaning of this terminology should not be confused with the meaning of "Modified Hankel functions" to be introduced later on Page 62.

The modified Bessel functions which we shall need have some function of depth as argument.

Up to this point we have shown various forms of solutions to Bessel's equation and have stated that if z is replaced by kz, k^2 being introduced as a separation constant of undetermined value, it may develop that k can take on only certain permitted values (eigenvalues) and still permit satisfying boundary conditions. Even so, there is a solution (eigenfunction) for each eigenvalue, and a linear combination of eigenfunctions is a solution. Thus, Eq. (1.104) as an example might more generally take the form

$$Z = \sum_{i=1}^{\infty} E_i H_o^{\,1}(k_i z) + F_i H_o^{\,2}(k_i z). \qquad (1.113)$$

If there are no restrictions on k, the summation is replaced by an integration over k.

Solutions of Equation for Depth Function

The following equation occurs in the study of sound propagation in stratified media

$$\frac{d^2\phi}{dz^2} + \left[k_o^{\,2}(1 + \beta z) - \alpha^2\right]\phi = 0. \qquad (1.114)$$

There are at least two approaches to its solution. The first approach which we shall describe reduces Eq. (1.114) to a Bessel's equation of order (1/3).

Eq. (1.114) is of the form

$$\frac{d^2\phi}{dz^2} + (Bz + C)\phi = 0. \qquad (1.115)$$

We let

$$Bz + C = (\tfrac{3}{2}B\xi)^{2/3} \tag{1.116}$$

and

$$\phi(z) = (Bz + C)^{1/2}V(\xi) = (\tfrac{3}{2}B\xi)^{1/3}V(\xi). \tag{1.117}$$

Then

$$\xi = \frac{2}{3B}(Bz + C)^{3/2}, \tag{1.118}$$

and

$$\frac{d\xi}{dz} = (Bz + C)^{1/2} = (\tfrac{3}{2}B\xi)^{1/3}. \tag{1.119}$$

We then find

$$\frac{d\phi}{dz} = \frac{d}{d\xi}\left[(\tfrac{3}{2}B\xi)^{1/3}V(\xi)\right]\frac{d\xi}{dz}$$

and

$$\frac{d^2\phi}{dz^2} = \frac{d}{d\xi}\left\{\frac{d}{d\xi}\left[(\tfrac{3}{2}B\xi)^{1/3}V(\xi)\right]\frac{d\xi}{dz}\right\}\frac{d\xi}{dz}. \tag{1.120}$$

Substituting Eq. (1.116), Eq. (1.117, and Eq. (1.120) worked out, into Eq. (1.115), we obtain

$$\frac{d^2V}{d\xi^2} + \frac{1}{\xi}\frac{dV}{d\xi} + (1 - \frac{1}{9z^2})V = 0 \tag{1.121}$$

which is the desired form.

Expressing V as Hankel functions we have *

$$V = AH^{1}_{\frac{1}{3}}(\xi) + BH^{2}_{\frac{1}{3}}(\xi). \tag{1.122}$$

Finally, the full expression for ϕ in terms of Hankel functions involving z is

$$\phi = A\gamma H^{1}_{\frac{1}{3}}\left(\frac{2\gamma^{3}}{3K_{o}^{2}\beta}\right) + B\gamma H^{2}_{\frac{1}{3}}\left(\frac{2\gamma^{3}}{3K_{o}\beta}\right) \tag{1.123}$$

in which

$$\gamma = \left[k_{o}^{2}(1 + \beta z) - \alpha^{2}\right]^{\frac{1}{2}}. \tag{1.124}$$

The second procedure is to make the substitution

$$\eta = \frac{K_{o}^{2}(1 + \beta z) - \alpha^{2}}{(K_{o}^{2}\beta)^{2/3}}. \tag{1.125}$$

Substitution of Eq. (1.125) in Eq. (1.114) yields the Stokes equation

$$\frac{d^{2}\phi}{d\eta^{2}} + \eta\phi = 0. \tag{1.126}$$

Solutions to this equation may be found readily by expressing ϕ in a power series of ascending powers, substituting in Eq. (1.126), collecting coefficients of like powers and equating the collected coefficients of each power to zero. The coefficients $a_{(3n)}$, n = 1, 2, 3, ..., are all expressible in terms of a_{o}. This gives one series solution of the following form

$$f(\eta) = a_{o}\left(1 - \frac{1}{3!}\eta^{3} + \frac{1 \cdot 4}{6!}\eta^{6} - \frac{1 \cdot 4 \cdot 7}{9!}\eta^{9} + \ldots\right). \tag{1.127}$$

* See reference 6

The coefficients a_{3n+1}, n = 1,2,3,..., are all expressible in terms of a_1, giving a second solution

$$g(\eta) = a_1\left(\eta - \frac{2}{4!}\eta^4 + \frac{2\ 5}{7!}\eta^7 - \frac{2\ 5\ 8}{10!}\eta^{10} + \ldots\right) \tag{1.128}$$

Note that in both $f(\eta)$ and $g(\eta)$ the signs alternate but with negative argument all signs are alike in either function. All other coefficients, a_2, a_5 are zero.

The Airy functions, which have been tabulated, are related to f and g as follows:

$$Ai(-\eta) = \frac{3^{-\frac{2}{3}}}{\Gamma(\frac{2}{3})} f(\eta) - \frac{3^{-\frac{1}{3}}}{\Gamma(\frac{1}{3})} g(\eta) \tag{1.129}$$

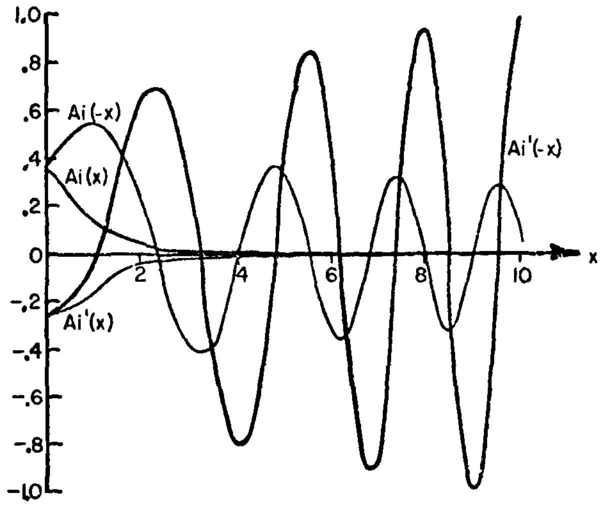
$$Bi(-\eta) = \sqrt{3}\left[\frac{3^{-\frac{2}{3}}}{\Gamma(\frac{2}{3})} f(\eta) + \frac{3^{-\frac{1}{3}}}{\Gamma(\frac{1}{3})} g(\eta)\right]. \tag{1.130}$$

The solution of Eq. (1.126) is expressible in terms of the Airy functions as

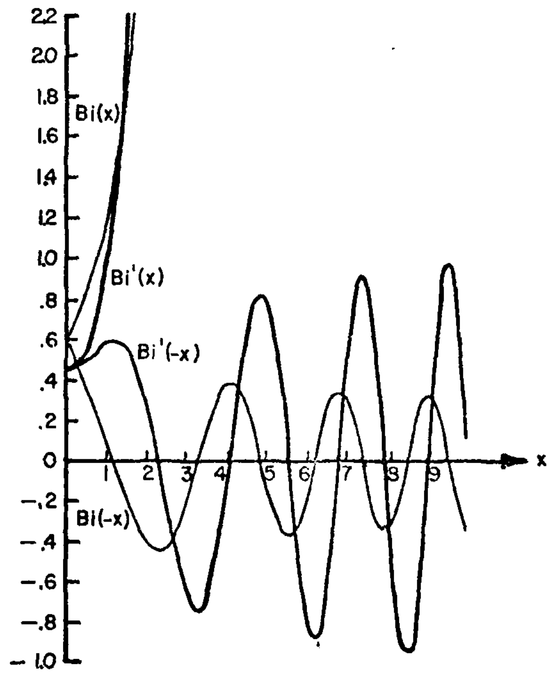
$$\phi = E\ Ai(-\eta) + F\ Bi(-\eta). \tag{1.131}$$

Finally, just as we combined Bessel functions of the first and second kind to form Hankel functions, so we may combine Airy functions to form modified Hankel functions

$$h_1 = 12^{1/6}\ e^{-i\pi/6}\ \left[Ai(-\eta) - i\ Bi(-\eta)\right], \tag{1.132}$$
$$h_2 = 12^{1/6}\ e^{+i\pi/6}\ \left[Ai(-\eta) + i\ Bi(-\eta)\right].$$



Ai(±x), Ai'(±x)



Bi(±x), Bi'(±x)

Fig. 1.14 and 1.15

Note that when η is positive, the sign in the terms for Ai and Bi alternate giving rise to oscillating functions. When η is negative, which can occur for example when β is negative and z is large, Ai($-\eta$) is a monatonically decreasing function with increasing $-\eta$ and Bi($-\eta$) is a monatomically increasing function with increasing $-\eta$.

The modified Hankel functions are expressible in terms of z by substituting Eq. (1.125) for η. The expression for ϕ is then

$$\phi = C\, h_1(\eta) + D\, h_2(\eta). \tag{1.133}$$

In contrast to the solution in regular Hankel functions, Eq. (1.123), there are constants as coefficients of h_1 and h_2 as compared to $\gamma(z)$ in the coefficients of $H^1_{\frac{1}{3}}$ and $H^2_{\frac{1}{3}}$. This makes a neater solution.

The Airy functions and their derivatives are plotted in Fig.1.14 and Fig.1.15 taken from Ref. 4.

LAPLACIAN

The wave equation will be derived from physical principles in the treatment of sound propagation. In Cartesian coordinates, it is

$$\nabla^2\, \Psi = \frac{1}{c^2}\, \frac{\partial^2 \Psi}{\partial t}. \tag{1.134}$$

The operator ∇^2 called the Laplacian needs to be defined. The definition will be in terms of Cartesian coordinates and we shall then show how to express it in any coordinate system.

The operator ∇ is defined in vector analysis as

$$\nabla = i\, \frac{\partial}{\partial x} + j\, \frac{\partial}{\partial y} + k\, \frac{\partial}{\partial z} \tag{1.135}$$

in which *i*, *j*, and *k* are unit vectors in the x, y, and z directions respectively. The dot product of ∇ by itself is

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{1.136}$$

The Eq. (1.136) defines ∇^2 in Cartesian coordinates. When ∇^2 operates on a scalar quantity Ψ, we have the left member of the wave equation, Eq. (1.134). Use of Eq. (1.136) in Eq. (1.134) gives the wave equation in Cartesian coordinates.

An expression for ∇^2 in generalized coordinates, q_1, q_2, and q_3, is derivable (e.g. see Ref. 5) and takes the following form:

$$\nabla^2 = h_1 h_2 h_3 \left[\frac{\partial}{\partial q_1}\left(\frac{h_1}{h_2 h_3}\frac{\partial}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_2}{h_3 h_1}\frac{\partial}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_3}{h_1 h_2}\frac{\partial}{\partial q_3}\right)\right]. \tag{1.137}$$

The h's are given by

$$h_i = \frac{\partial q_i}{\partial n} \tag{1.138}$$

in which n is the normal in the positive direction to the level surface in q_i, i.e., the surface obtained with q_i constant.

In spherical coordinates we have a radial distance r from the origin, the colatitude θ, and the longitude ϕ measured from the positive x axis toward the positive y axis. The level surface in r is a sphere. The normal to this is radial in direction and $\frac{\partial r}{\partial n} = h_1 = 1$.

The level surface in θ is a cone, and the normal to this is a vector normal to the cone giving $\frac{\partial \theta}{\partial n} = h_2 = \frac{1}{r}$. A level surface in ϕ is a plane and

$\frac{\partial\phi}{\partial n} = h_3 = \frac{1}{r \sin\theta}$. Using these h's, Eq. (1.137) for the Laplacian becomes

$$\nabla^2 = \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial r}(r^2 \sin\theta \frac{\partial}{\partial r}) + \frac{\partial}{\partial\theta}(\sin\theta \frac{\partial}{\partial\theta}) + \frac{\partial}{\partial\phi}(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}). \quad (1.139)$$

For a spherical wave in which spheres are equiphase surfaces of constant amplitude, this reduces to

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}. \quad (1.140)$$

A third system of coordinates with which we should have familiarity is that of cylindrical coordinates. Here the three coordinates are r, the radial distance from the polar axis; z, the distance from the origin along the polar axis; and θ, the longitude measured from the positive x axis. The level surface in r is a cylinder, the normal is radial and $\frac{\partial r}{\partial n} = h_1 = 1$. The level surface in z is a plane normal to the polar axis, the normal is along the polar (z) axis, and $\frac{\partial z}{\partial n} = h_2 = 1$. The level surface in θ is a plane including the polar axis, the normal is horizontal and tangent to a cylinder of radius r about the polar axis and $\frac{\partial\theta}{\partial n} = h_3 = \frac{1}{r}$.

Applying the general Eq. (1.138) with these h values, we obtain the Laplacian

$$\nabla^2 = \frac{1}{r}\left[\frac{\partial}{\partial r}(r \frac{\partial}{\partial r}) + r \frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial^2}{\partial\theta^2}\right]$$

$$= \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \quad (1.141)$$

For cylindrical symmetry in which a sound wave is traveling inward or outward with equiphase surfaces which are infinite cylinders, this reduces to

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\, r\, \frac{\partial}{\partial r} = \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}. \qquad (1.142)$$

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